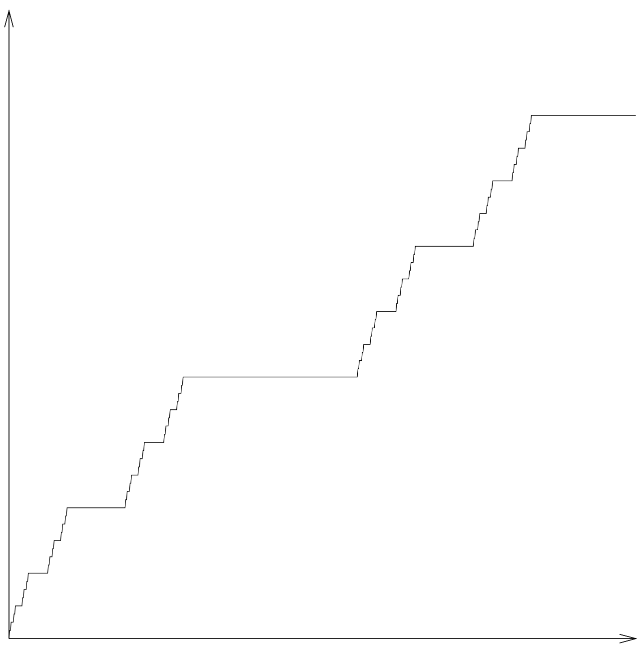
Metric Preserving Functions

by

Jozef Doboš



Jozef Doboš

Metric Preserving Functions

Preface

We call a function $f : [0, +\infty) \to [0, +\infty)$ *metric preserving* iff for each metric space (X, d) the function $f \circ d$ is a metric on X.
Although the first reference in the literature to the notion of metric preserving functions seems to be [67], the first detailed study of these functions was by Sreenivasan in 1947 [58]. Kelley's classic text in general topology [33] mentions some of early known results in an excercise. This is the only place where this topic is treated in a widely-known monograph. In the past two decades, a significant literature has developed on the subject of metric preserving functions.
The purpose of this booklet is to present some of results and techniques of the field to a broader mathematical audience.

Technical University, Košice J. D.

Research supported by VEGA 2/5124/98

This book was typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX, the macro system of the American Mathematical Society

We have reproduced certain sections from the survey by Paul Corazza [15] for the sake of expository clarity.

0. Introduction

We begin with three examples:

Example 1. You are in New York City, just got a cab – you get in. The meter shows \$ 1.50. This is a flat rate, as the driver says. Now every $\frac{1}{8}$ of a mile will cost you \$ 0.25. The graph is shown in Figure 1.

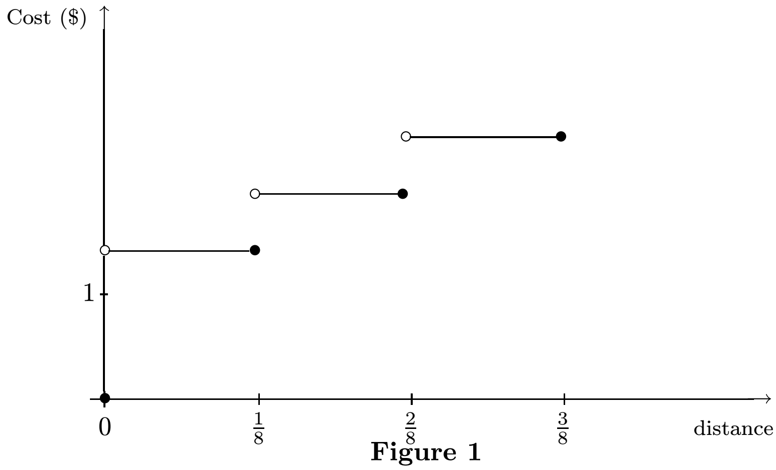


Figure 1

Example 2. Have you called your friend in Paris, France from NY? The current AT&T full rate is \$ 1.71 for the first minute and \$ 1.08 for every additional minute. This data is shown on the graph in Figure 2.

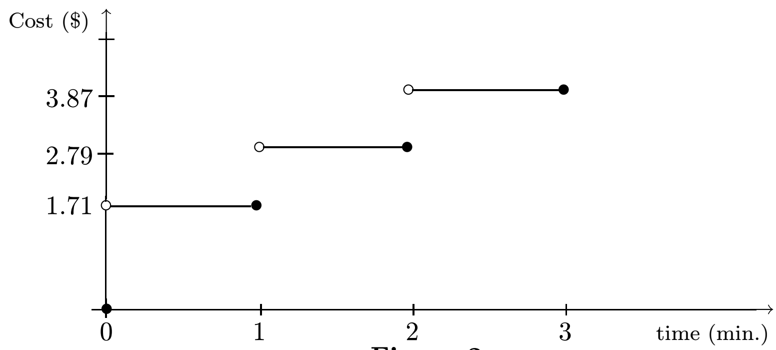


Figure 2

Example 3. Imagine that you operate a truck fleet. Observe that the distance travelled by a truck corresponds to a certain amount of money spent on fuel. Conservative estimates show about 10 miles per gallon. Assume 1 gallon of fuel costs \$ 1.00.

If we place the distance (in miles) on the x-axis and the cost (in \$) on the y-axis then the function $f : [0, +\infty) \to [0, +\infty)$ is given by $f(x) = \lceil \frac{x}{10} \rceil$, $x \geq 0$; where $x \mapsto \lceil x \rceil$ is the *ceiling function*, which returns the smallest integer that is not less than its argument, i.e. $\lceil a \rceil = \min([a, +\infty) \cap \mathbb{Z})$.

What do these three examples have in common? In each instance we replace the actual distance, or time, as in Example 2, by the cost.

Let X be a nonempty set. We say that a function $d : X^2 \to [0, +\infty)$ is a *metric*, if the following axioms are met for each $x, y, z \in X$:

(M1) $d(x, y) = 0$ iff $x = y$,

(M2) $d(x, y) = d(y, x)$,

(M3) $d(x, y) \leq d(x, z) + d(z, y)$ (the triangle inequality).

The pair (X, d) is called a *metric space*. It is essentially a set in which it is possible to speak of the distance between each two of its elements.

Formalizing the above discussion we have the following general problem:

Given a metric d, we shall refer to d as 'the old metric'. We will consider functions $f : [0, +\infty) \to [0, +\infty)$ such that the composition ζ, defined by:

$$\zeta(x, y) = f(d(x, y))$$

is a 'new' metric.

Probably the first example which comes to mind is the ceiling function $x \mapsto \lceil x \rceil$. In fact, such a composition will produce a metric (see the following theorem).

Theorem 1. (See Kelley [33], p. 131.) *Let f be a real-valued function defined for nonnegative numbers, and such that f is continuous, (the continuity of f is needed only for the equivalence of d with ζ, see Theorem 3.2), nondecreasing and satisfying the following two conditions:*

(1) $f(a) = 0 \Leftrightarrow a = 0$, *and*

(2) $f(a + b) \leq f(a) + f(b)$, *for each* a, b.

Let (X, d) be a metric space and let $\zeta(x, y) = f(d(x, y))$ for each $x, y \in X$. Then (X, ζ) is a metric space and the metrics d and ζ are topologically equivalent.

Corollary 1. *Let* (X,d) *be a metric space and let*

$$\zeta(x,y) = \lceil d(x,y) \rceil \text{ for each } x,y \in X.$$

Then (X,ζ) *is a metric space.*

Proof. Let $f : [0,+\infty) \to [0,+\infty)$ be defined by $f(x) = \lceil x \rceil$. (See Fig. 3.) Let $a, b \geq 0$. Since $\lceil a \rceil \in \mathbb{Z}$, $\lceil a \rceil \geq a$, and $\lceil b \rceil \in \mathbb{Z}$, $\lceil b \rceil \geq b$, we have

$$\lceil a \rceil + \lceil b \rceil \in \mathbb{Z}, \quad \lceil a \rceil + \lceil b \rceil \geq a + b.$$

Thus

$$\lceil a \rceil + \lceil b \rceil \in [a+b, +\infty) \cap \mathbb{Z},$$

which yields

$$\lceil a + b \rceil = \min([a+b,+\infty) \cap \mathbb{Z}) \leq \lceil a \rceil + \lceil b \rceil.$$

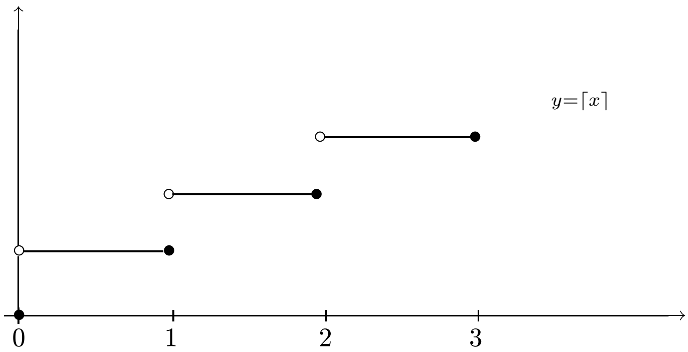


Figure 3

Corollary 2. *Let* (X,d) *be a metric space and let*

$$\zeta(x,y) = \frac{d(x,y)}{1+d(x,y)} \text{ for each } x,y \in X.$$

Then (X,ζ) *is a metric space and the metrics* d *and* ζ *are topologically equivalent.*

Proof. Let $f : [0,+\infty) \to [0,+\infty)$ be defined by (see Fig. 4)

$$f(x) = \frac{x}{1+x}, \quad (x \geq 0).$$

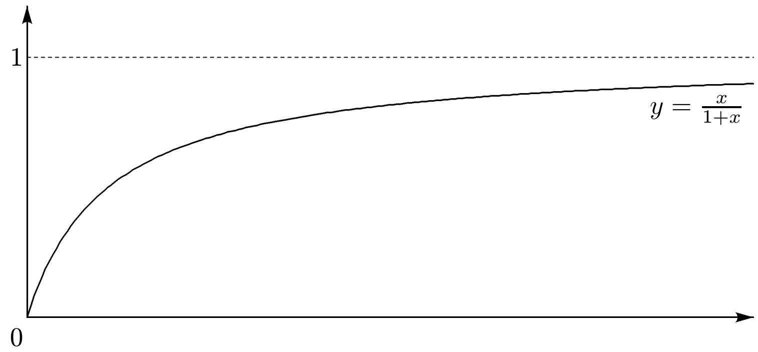


Figure 4

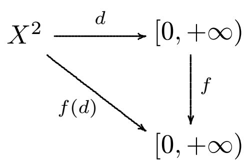
Let $a, b \geq 0$. Then

$$f(a+b) = \frac{a+b}{1+a+b} = \frac{a}{1+a+b} + \frac{b}{1+a+b} \leq \frac{a}{1+a} + \frac{b}{1+b} = f(a)+f(b).$$

1. Preliminaries

Let (X, d) be a metric space. For each $f : [0, +\infty) \to [0, +\infty)$ define a function $d_f : X^2 \to [0, +\infty)$ as follows

$$d_f(x, y) = f(d(x, y)) \text{ for each } x, y \in X.$$



We call a function $f : [0, +\infty) \to [0, +\infty)$ *metric preserving* iff for each metric space (X, d) the function d_f is a metric on X. For example, we can derive a bounded metric from a given metric by the function $x \mapsto \frac{x}{1+x}$ (see Corollary 0.2). This idea is used in the construction of the Fréchet metric on a product of a countable family of metric spaces, i.e.

$$\varrho(x, y) = \sum_{i=1}^{\infty} 2^{-i} \cdot \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)}.$$

Denote by $\mathcal{O}$ the set of all functions $f : [0, +\infty) \to [0, +\infty)$ with

$$f^{-1}(0) = \{0\}.$$

We call such functions *amenable*. It is easy to see that every metric preserving function is amenable.

Let us recall that a function $f : [0, +\infty) \to [0, +\infty)$ is said to be *subadditive* if it satisfies the inequality

$$f(x + y) \leqq f(x) + f(y)$$

whenever $x, y \in [0, +\infty)$. (See [31] and [53].)

In the following we show an importance of subadditivity in our investigations. This theorem first appeared in Wilson's early paper [67].

Wilson's Theorem. *Let $f \in \mathcal{O}$ be such that*

$$a, b, c \geq 0 \text{ and } a \leq b + c \text{ imply}$$
$$f(a) \leq f(b) + f(c).$$

Then f is metric preserving.

This theorem can be expressed somewhat differently. (See Theorem 0.1.)

Theorem 1. *Suppose that $f \in \mathcal{O}$ is nondecreasing and subadditive. Then f is metric preserving.*

Proof. Let (X, d) be a metric space; we show that $f \circ d$ is a metric. Properties (M1) and (M2) are easy to check. For (M3), let $x, y, z \in X$, and let

$$a = d(x, z), b = d(z, y), \text{ and } c = d(x, y).$$

It suffices to show that $f(a) + f(b) \geq f(c)$. But

$$\begin{aligned} f(a) + f(b) &\geq f(a + b) && \text{(subadditive)} \\ &\geq f(c) && \text{(nondecreasing),} \end{aligned}$$

as required.

On the other hand, the next proposition provides a necessary condition for a function to be metric preserving.

Proposition 1. *Every metric preserving function is subadditive.*

Proof. Let $f : [0, +\infty) \to [0, +\infty)$ be a metric preserving function. Denote by e the usual metric on the real line, i.e. $e(x, y) = |x - y|$ for each $x, y \in \mathbb{R}$. Suppose that $a, b \in [0, \infty)$. Then

$$f(a + b) = e_f(0, a + b) \leq e_f(0, a) + e_f(a, a + b) = f(a) + f(b).$$

The following criterion of subadditivity is well known.

Proposition 2. *Let $f \in \mathcal{O}$ and the function $x \mapsto \frac{f(x)}{x}$ be nonincreasing on $(0, +\infty)$. Then f is subadditive.*

Proof. Let $a, b \in (0, +\infty)$. Then

$$f(a + b) = a \cdot \frac{f(a + b)}{a + b} + b \cdot \frac{f(a + b)}{a + b} \leq a \cdot \frac{f(a)}{a} + b \cdot \frac{f(b)}{b} = f(a) + f(b).$$

While subadditivity is an important necessary condition, the following three examples show that it is not sufficient for an amenable function to be metric preserving.

Example 1. (See [15].) Define $f : [0, +\infty) \to [0, +\infty)$ as follows

$$f(x) = \frac{x}{1 + x^2} \text{ for all } x \in [0, +\infty).$$

Since the function $x \mapsto \frac{f(x)}{x}$ is nonincreasing on $(0, +\infty)$, by Proposition 2 the function f is subadditive (see Fig. 5). Note that f is not metric preserving. (See Corollary 2.1.)

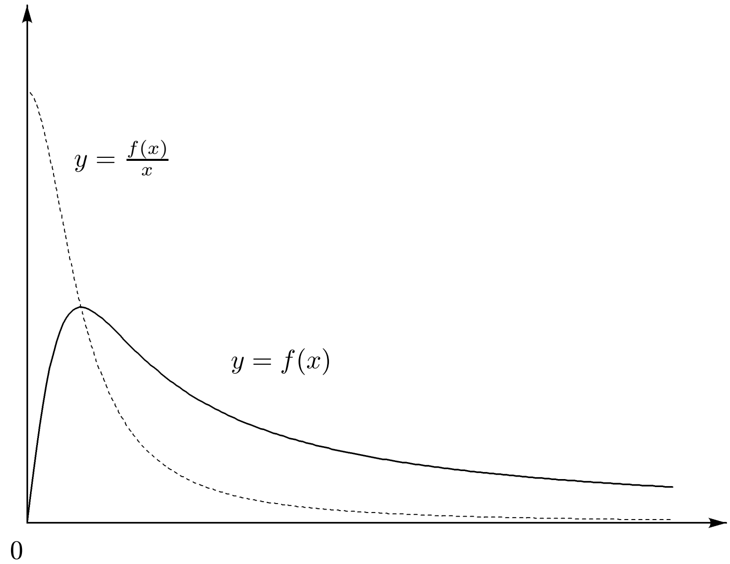


Figure 5

Example 2. (See [25], p.133.) Define $f : [0, +\infty) \to [0, +\infty)$ as follows

$$f(x) = \begin{cases} x & \text{if } x < 1, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

By Proposition 2 the function f is subadditive. (See Fig. 6.) Note that this function is not metric preserving. (See Theorem 3.1.)

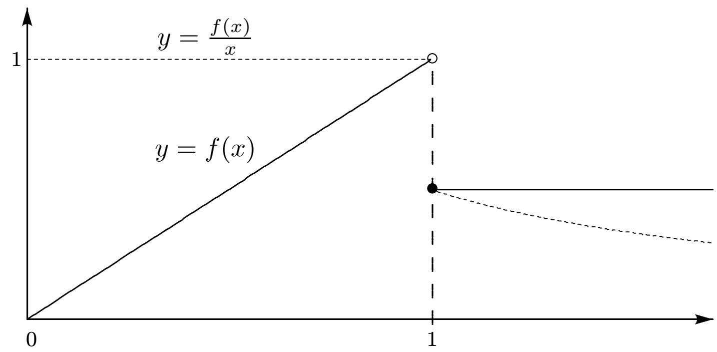


Figure 6

Example 3. Define $f : [0, +\infty) \to [0, +\infty)$ as follows

$$f(x) = \begin{cases} \frac{x}{1+x} & \text{on } \mathbb{Q}, \\ 1 & \text{otherwise.} \end{cases}$$

It is not difficult to verify that f is subadditive. Evidently f is discontinuous at each point of $[0, +\infty)$. Note that f is not metric preserving.

The converse of Proposition 2 is not true, as the following example shows.

Example 4. Define $f : [0, +\infty) \to [0, +\infty)$ as follows

$$f(x) = \tfrac{3}{4} \cdot x + \tfrac{1}{2} - \tfrac{1}{4} \cdot |3x - 1| + \tfrac{1}{4} \cdot |3x - 2| - \tfrac{3}{4} \cdot |x - 1|.$$

Since the function f is metric preserving, it is subadditive. (See Fig. 7.) Note that the function $x \mapsto \frac{f(x)}{x}$ is not nonincreasing.

Simple examples of metric preserving functions are concave functions. (See [56], [4], and [58].) Let us recall that a function $f : [0, +\infty) \to [0, +\infty)$ is called *concave* iff

$$f((1-t)x_1 + tx_2) \geq (1-t)f(x_1) + tf(x_2),$$

whenever $x_1, x_2 \in [0, +\infty)$ and $0 \leq t \leq 1$.

A less restrictive definition of a concave function f is the requirement that f be *midpoint-concave*: "at the midpoint of an interval the curve lies above the chord", i.e. the inequality

$$f\left(\frac{x+y}{2}\right) \geq \frac{f(x)+f(y)}{2}$$

holds for all $x, y \in [0, +\infty)$. Note that for amenable functions those two notions are equivalent. (See [26].)

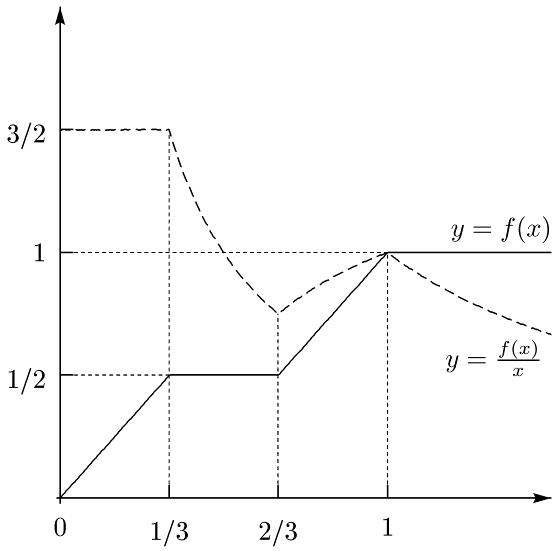


Figure 7

Lemma 1. *Suppose that $f \in \mathcal{O}$ is concave. Then the function $x \mapsto \frac{f(x)}{x}$ is nonincreasing on $(0, +\infty)$.*

Proof. Let $a, b \in (0, +\infty)$, $a < b$. Put $t = \frac{a}{b}$, $x_1 = 0$, $x_2 = b$. Since f is concave, we have $f(a) \geq \frac{a}{b} \cdot f(b)$. Therefore the function $x \mapsto \frac{f(x)}{x}$ is nonincreasing on $(0, +\infty)$.

Theorem 2. *Let $f \in \mathcal{O}$. Then f is concave iff*

(*) $\quad \forall t \geq 0 \; \forall x, y, z \in [0, t], x + t = y + z : f(x) + f(t) \leq f(y) + f(z).$

Proof. Suppose that f is concave. Let $t \geq 0$, $x, y, z \in [0, t]$, $x + t = y + z$. Suppose that $y \leq z$. We distinguish two cases.

1) Let $y = z$. Since $y = \frac{1}{2}(x + t)$, by the assumption we have

$$f(y) \geq \tfrac{1}{2}(f(x) + f(t)), \text{ i.e. } f(x) + f(t) \leq f(y) + f(y) = f(y) + f(z).$$

2) Let $y < z$. Let $p > 0$ be such that $y = px + (1 - p)z$. Then $z = pt + (1 - p)y$. By the assumption we obtain

$$\begin{aligned} f(y) &\geq pf(x) + (1 - p)f(z), \\ f(z) &\geq pf(t) + (1 - p)f(y), \end{aligned}$$

Thus $f(y) + f(z) \geq p(f(x) + f(t)) + (1 - p)(f(z) + f(y))$, which yields $f(x) + f(t) \leq f(y) + f(z)$.

On the other hand, suppose that (*) holds. We show that f is midpoint-concave. Let $0 < x < y$. Since $0 < x < \frac{1}{2}(x + y) = \frac{1}{2}(x + y) < y$, by (*) we have $f(x) + f(y) \leq f\left(\frac{x+y}{2}\right) + f\left(\frac{x+y}{2}\right)$.

Remark. Observe that putting $x = 0$ in (*) we obtain the subadditivity of f.

Theorem 3. *Suppose that $f \in \mathcal{O}$ is concave. Then f is metric preserving.*

Proof. The subadditivity of f follows from Lemma 1 (or from Remark).

Now, we will show that f is nondecreasing. We procede by contradiction. Suppose that there are $x, y \in (0, +\infty)$ such that $x < y$ and $f(x) > f(y)$. Put

$$t = \frac{f(y)}{f(x)}, \quad x_1 = \frac{yf(x) - xf(y)}{f(x) - f(y)}, \quad x_2 = x.$$

Since f is concave, we have

$$f(y) \geq (1 - t)f(x_1) + f(y),$$

which yields $f(x_1) \leq 0$, a contradiction.

A continuous, nondecreasing, metric preserving function which is not concave is shown in Example 4.

The next comparison test of subadditivity is a generalization of Proposition 2.

Proposition 3. (See [37].) *Let* $f, g \in \mathcal{O}$. *If* g *is subadditive and the function* $x \mapsto \frac{f(x)}{g(x)}$ *is nonincreasing on* $(0, +\infty)$, *then* f *is subadditive.*

Proof. For $a, b > 0$ we have

$$\begin{aligned} f(a+b) &= \frac{f(a+b)}{g(a+b)} \cdot g(a+b) \leq \frac{f(a+b)}{g(a+b)} \cdot (g(a)+g(b)) \\ &\leq \frac{f(a)}{g(a)} \cdot g(a) + \frac{f(b)}{g(b)} \cdot g(b) = f(a) + f(b). \end{aligned}$$

Let us finally note that subadditivity admits a nice characterization in terms of infimal convolution. If $f, g \in \mathcal{O}$, then their *infimal convolute* $f \Box g$ (see [42] and [59]) is the function that sends each $x \in [0, +\infty)$ to the real number

$$(f \Box g)(x) = \inf\{f(y) + g(z) : \ y, z \in [0, +\infty) \text{ and } y + z = x\}.$$

Proposition 4. (See [59].) *Let* $f, g \in \mathcal{O}$. *Then the following statements hold:*

(1) f *is subadditive iff* $f \Box f = f$,
(2) *if* $\min(f, g)$ *is subadditive, then* $f \Box g = \min(f, g)$, *and*
(3) *if* f *and* g *are both subadditive, then* $f \Box g$ *is the largest subadditive minorant of* $\min(f, g)$.

Remark. Let $f, g \in \mathcal{O}$ be nondecreasing. Suppose that $x \mapsto \frac{f(x)}{x}$ and $x \mapsto \frac{g(x)}{x}$ are nonincreasing on $(0, +\infty)$. (This is the case for instance if f and g are concave.) Put $h = \min(f, g)$. Then h is nondecreasing and $x \mapsto \frac{h(x)}{x}$ is nonincreasing on $(0, +\infty)$. Consequently, h is subadditive. Thus $h = f \Box g$.

For recent results on subadditive functions, see [39] and [40]. Let us mention only one of the results proved there.

Proposition 5. (See [40].) *Every subadditive and right-continuous bijection of* $[0, +\infty)$ *is a homeomorphism.*

2. Characterization of metric preserving functions

Let a, b and c be positive real numbers. We call the triplet (a,b,c) a *triangle triplet* (see [60]) iff

$$a \leq b+c,\ \ b \leq a+c,\ \text{ and } c \leq a+b;$$

equivalently,
$$|a-b| \leq c \leq a+b;$$

i. e.
$$a+b+c \geq 2 \cdot \max\{a,b,c\}.$$

Triangle triplets are used in place of the more awkward terms $d(x,y)$, $d(x,z)$, $d(z,y)$ for various metrics d. The following result gives a characterization of triangle triplets, which is based on the fact that each three-points metric space has a representation by certain subspace of the Euclidean plane. (See [4].)

Proposition 1. *Let a, b and c be positive real numbers. Then the triplet (a,b,c) is a triangle triplet iff there are $x,y,z \in \mathbb{R}^2$, $x \neq y \neq z \neq x$, such that*

$$a = e(x,y), \quad b = e(x,z), \quad c = e(z,y),$$

where e denotes the Euclidean metric on $\mathbb{R}^2$.

Proof. Suppose that (a,b,c) is a triangle triplet. Put

$$x = \left(\frac{a}{2},0\right), \quad y = \left(-\frac{a}{2},0\right),$$
$$z = \left(\frac{c^2-b^2}{2a}, \frac{1}{2a} \cdot \sqrt{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}\right).$$

Then $a = e(x,y)$, $b = e(x,z)$, $c = e(z,y)$.
On the other hand, if $x,y,z \in \mathbb{R}^2$, $x \neq y \neq z \neq x$, then

$$(e(x,y), e(x,z), e(z,y)) \text{ is a triangle triplet.}$$

This is immediate from the triangle inequality.

As a corollary we obtain the following theorem which gives a characterization of metric preserving functions. (See [58], [4], and [17].)

Theorem 1. *Let $f \in \mathcal{O}$. Then the following are equivalent:*

(1) *f is metric preserving,*
(2) *if (a, b, c) is a triangle triplet, then so is $(f(a), f(b), f(c))$,*
(3) *if (a, b, c) is a triangle triplet, then $f(a) \leq f(b) + f(c)$, and*
(4) $\forall x, y \in [0, +\infty) :\ \max\{f(z);\ |x - y| \leq z \leq x + y\} \leq f(x) + f(y)$.

The following corollary shows that no function $f \in \mathcal{O}$ having the x-axis as a horizontal asymptote is metric preserving.

Corollary 1. (See [4].) *Let f be a metric preserving function. Then*

$$\forall a, b \in [0, +\infty) :\ a \leq 2b \Rightarrow f(a) \leq 2f(b).$$

Note that the asumption "metric preserving" in Corollary 1 cannot be replaced by the assumption "subadditive", as Example 1.1 shows.

The proofs of the following two propositions are straightforward and we omit them.

Proposition 2. (See [4].)

(1) *If f, g are metric preserving and $k > 0$, then each of $f \circ g$, $f + g$, $k \cdot f$, and $\max(f, g)$ is metric preserving.*
(2) *If f_n ($n \in \mathbb{N}$) are metric preserving functions that converge to a function $f \in \mathcal{O}$, then f is metric preserving. Under the same hypothesis, if $\sum_{n=1}^{\infty} f_n$ converges to a function s, then s is metric preserving.*
(3) *If $(f_t)_{t \in T}$ is any indexed family of metric preserving functions that is pointwise bounded, then the function $x \mapsto \sup\{f_t(x);\ t \in T\}$ is metric preserving.*

Let Ω denote the first uncountable ordinal number. A transfinite sequence $(a_\xi)_{\xi < \Omega}$ of nonnegative reals is said to be convergent and have a limit $a \in [0, +\infty)$ if for each $\varepsilon > 0$ there exists an ordinal number $\alpha < \Omega$ such that $d(a_\xi, a) < \varepsilon$ whenever $\alpha \leq \xi < \Omega$. If $(a_\xi)_{\xi < \Omega}$ has a limit a, we write $\lim\limits_{\xi \to \Omega} a_\xi = a$.

A transfinite sequence $(f_\xi)_{\xi < \Omega}$ of functions $f_\xi : [0, +\infty) \to [0, +\infty)$ is said to be convergent and to have a limit function $f : [0, +\infty) \to [0, +\infty)$ if for each $x \in [0, +\infty)$ we have $\lim\limits_{\xi \to \Omega} f_\xi(x) = f(x)$.

Proposition 3. (See [4].) *If* $(f_\xi)_\xi < \Omega$ *is a tranfinite sequence of metric preserving functions which converges to a function* f*, then* f *is metric preserving.*

Finally, we will show that "most" metric preserving functions are not continuous. We call an amenable function f *tightly bounded* if for some $a > 0$, $f(x) \in [a, 2a]$ for all $x > 0$. (See [15].)

Proposition 4. *If* f *is amenable and tightly bounded, then* f *is metric preserving.*

Proof. Let $a > 0$ be such that for all $x > 0$, $f(x) \in [a, 2a]$, and let (a, b, c) be a triangle triplet. Then $f(a) \leq 2a = a + a \leq f(b) + f(c)$.

Every amenable, tightly bounded function is necessarily discontinuous at 0. It follows that there are $2^{\mathfrak{c}}$ tightly bounded, amenable functions (where $\mathfrak{c}$ is the cardinality of $\mathbb{R}$).

The following example shows that there is a metric preserving function which is nowhere continuous and nowhere of bounded variation.

Example 1. Define $f : [0, +\infty) \to [0, +\infty)$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \text{ is irrational}, \\ 2 & \text{otherwise.} \end{cases}$$

Since f is amenable and tightly bounded, it is metric preserving; because the sets $(0, +\infty) \cap \mathbb{Q}$ and $(0, +\infty) - \mathbb{Q}$ are dense in $(0, +\infty)$, f satisfies the required pathologies.

3. Strongly metric preserving functions

We call $f : [0, +\infty) \to [0, +\infty)$ *strongly metric preserving* if for all metric spaces (X, d), d_f is a metric topologically equivalent to d. (See [15].)

In this section we characterize the strongly metric preserving functions. An important theme here is the significance of the behavior of a metric preserving function at 0. We show that such an f is strongly metric preserving if and only if f is continuous at 0.

We begin with the following definition. Given a point x of a metric space (X, d) and a positive real number ε, the *open ball* with the center x and radius ε is the set

$$B_d(x; \varepsilon) = \{y \in X;\ d(x, y) < \varepsilon\}.$$

If in Theorem 2.1 we let $c = |a - b|$, we obtain the following result.

Proposition 1. (See [7].) *If f is metric preserving, then*

$$\forall a, b \in [0, +\infty) :\ |f(a) - f(b)| \leq f(|a - b|).$$

Theorem 1. (See [4].) *Supose that f is metric preserving. Then the following are equivalent:*

(1) *f is continuous,*
(2) *f is continuous at 0, and*
(3) $\forall \varepsilon > 0\ \exists x > 0 :\ f(x) < \varepsilon.$

Proof. It follows from Proposition 1 that (2) implies (1). Indeed, let $\varepsilon > 0$. From the continuity of f at 0 it follows that there is $\delta > 0$ such that

$$x \in [0, \delta) \text{ implies } f(x) < \varepsilon,$$

which yields

$$a, b \in [0, +\infty), \text{ and } |a - b| \leq \delta \text{ implies } |f(a) - f(b)| \leq f(|a - b|) < \varepsilon.$$

It follows from Corollary 2.1 that (3) implies (2). Indeed, let $\varepsilon > 0$. Then there is $x_0 > 0$ such that $f(x_0) < \frac{\varepsilon}{2}$. Put $\delta = 2x_0$. By Corollary 2.1 we have

$$x \in [0, \delta] \text{ implies } f(x) \leq 2 \cdot f(x_0) < \varepsilon.$$

Note that the asumption "metric preserving" in Theorem 1 cannot be replaced by the assumption "subadditive", as Example 1.2 shows.

Corollary 1. *Let f be metric preserving. If f is discontinuous, there is $\varepsilon > 0$ such that for all $x > 0$, $f(x) > \varepsilon$.*

Let us recall that a metric space (X, d) is *topologically discrete*, iff for every x in X there is $\varepsilon > 0$ such that $B_d(x; \varepsilon) = \{x\}$. We say that a metric space (X, d) is *uniformly discrete*, iff there is $\varepsilon > 0$ such that $d(x, y) > \varepsilon$ for each $x, y \in X$, $x \neq y$.

Proposition 2. *Let f be metric preserving. Then the following are equivalent:*

(1) *f is discontinuous;*
(2) *(X, d_f) is an uniformly discrete metric space for every metric space (X, d).*

Proof. It follows from Corollary 1 that (1) implies (2).

Now we will show that (2) implies (1). Let e denote the ususal metric on the real line. Let $\varepsilon > 0$ be such that $e_f(x, y) > \varepsilon$ whenever $x, y \in \mathbb{R}$, $x \neq y$. Then for each $a \in (0, +\infty)$ we have $\varepsilon < e_f(a, 0) = f(a)$.

Let us recall that two metrics ϱ and σ on a space X are *topologically equivalent* iff for each x in X and each $\varepsilon > 0$ there is $\delta > 0$ such that

$$B_\varrho(x; \delta) \subset B_\sigma(x; \varepsilon), \quad \text{and} \quad B_\sigma(x; \delta) \subset B_\varrho(x; \varepsilon).$$

We say that two metrics ϱ and σ on a space X are *uniformly equivalent* iff for each $\varepsilon > 0$ there is $\delta > 0$ such that for all $x, y \in X$ we have

$$\varrho(x, y) < \delta \text{ implies } \sigma(x, y) < \varepsilon, \quad \text{and} \quad \sigma(x, y) < \delta \text{ implies } \varrho(x, y) < \varepsilon.$$

The next theorem first appeared in [4]; one direction of the theorem was observed in Sreenivasan's early paper [58].

Theorem 2. (See [4].) *Suppose f is metric preserving. Then f is strongly metric preserving if and only if f is continuous.*

Proof. One direction follows from Proposition 2. For the other direction, suppose f is metric preserving and continuous. Let (X, d) be a metric space. We show that d_f and d are uniformly equivalent metrics. Let $\varepsilon > 0$. By continuity of f at 0, let $\delta > 0$ be such that for all $a \in [0, \delta)$, $f(a) < \varepsilon$. But now for each $x, y \in X$ we obtain

$$d(x, y) < \delta \text{ implies } d_f(x, y) < \varepsilon.$$

By Corollary 2.1 for each $a \in [0, +\infty)$ we have

$$f(a) < \frac{f(\varepsilon)}{2} \text{ implies } a < \frac{\varepsilon}{2}.$$

Put $\delta = \dfrac{f(\varepsilon)}{2} > 0$. Then for each $x, y \in X$ we obtain

$$d_f(x, y) < \delta \text{ implies } d(x, y) < \frac{\varepsilon}{2} < \varepsilon.$$

Theorem 3. (See [4].) *Let f be metric preserving. Suppose that (X, d) is a metric space which is not topologically discrete. Then the metrics d_f and d are topologically equivalent iff f is continuous.*

Proof. Suppose that the metrics d_f and d are topologically equivalent. Let $\varepsilon > 0$. Since (X, d) is not topologically discrete, there is $a \in X$ such that for each $\eta > 0$ there is $y \in B_d(a; \eta)$ such that $y \neq a$. Let $\delta > 0$ be such that $B_d(a; \delta) \subset B_{d_f}(a; \varepsilon)$. Choose $b \in B_d(a; \delta)$ such that $b \neq a$. Put $x = d(a, b) > 0$. Then $f(x) = d_f(a, b) < \varepsilon$.

If f is continuous, by Theorem 2 we obtain that the metrics d_f and d are topologically equivalent.

Theorem 4. (See [4].) *Let f be metric preserving. Suppose that (X, d) is a metric space which is topologically discrete. Then the metrics d_f and d are topologically equivalent.*

Proof. If f is discontinuous, by Proposition 2 we obtain that (X, d_f) is uniformly discrete, which yields that the metrics d_f and d are topologically equivalent.

If f is continuous, by Theorem 3 we obtain that the metrics d_f and d are topologically equivalent.

By the similar argumentation one can prove the following two theorems.

Theorem 5. (See [4].) *Let f be metric preserving. Suppose that (X, d) is a metric space which is not uniformly discrete. Then the metrics d_f and d are uniformly equivalent iff f is continuous.*

Theorem 6. (See [4].) *Let f be metric preserving. Suppose that (X, d) is a metric space which is uniformly discrete. Then the metrics d_f and d are uniformly equivalent.*

Example 1. Denote by e the usual metric on the set $X = \{n^{-1} : n \in \mathbb{N}\}$. Suppose that f is metric preserving function which is discontinuous. Then the metrics e_f and e are topologically equivalent, but they are not uniformly equivalent.

The results in this section show that, while the variety of possible metric preserving functions yields of a rich class of functions, from a strictly topological point of view the class is somewhat uninteresting. For any metric space (X, d), the number of possible distinct (up to homeomorphism) topologies that can be generated by the metrics d_f, as f ranges over the metric preserving functions, is ≤ 2 (d_f must either be equivalent to d or must induce the discrete topology on X). Nonetheless, the variety of distinct (up to isometry) metrics that can be generated in this way and that are topologically equivalent to the original metric — there are $\mathfrak{c}$ distinct metrics for each metric space (X, d) having two or more points — can lead to interesting results, as the theorem of M. Jůza (see [32]) shows.

4. Metric preserving functions and convexity

Before stating the result, we recall that a function $f : [0, +\infty) \to [0, +\infty)$ is called *convex* over $[r, s]$ iff

$$f((1-t)x_1 + tx_2) \leq (1-t)f(x_1) + tf(x_2) \tag{1}$$

whenever $x_1, x_2 \in [r, s]$ and $0 \leq t \leq 1$.

Moreover, f is *strictly convex* if $\leq$ is replaced by $<$ in (1). Convexity of f is equivalent to the assertion that for all $x, y \in [r, s]$, every point on the chord from $(x, f(x))$ to $(y, f(y))$ is above the graph of f in $[0, +\infty)^2$.

Lemma 1. (See [4].) *Suppose $f : [0, +\infty) \to [0, +\infty)$ is subadditive. Then for all positive integers n and all $x \in [0, +\infty)$,*

$$f(nx) \leq nf(x) \quad \textit{and} \quad 2^{-n}f(x) \leq f(2^{-n}x).$$

Proof. By induction.

Theorem 1. (See [7].) *Let $f : [0, +\infty) \to [0, +\infty)$ be metric preserving and $h > 0$. If f is convex on $[0, h]$, then f is linear on $[0, h]$.*

Proof. From the convexity we obtain

$$\forall a, b \in (0, h] : \ a \leq b \Rightarrow \frac{f(a)}{a} \leq \frac{f(b)}{b}. \tag{$*$}$$

We shall show that

$$f(x) = \frac{f(h)}{h} \cdot x \ \text{ for each } \ x \in [0, h].$$

Let $x \in (0, h]$. Choose a positive integer n such that $2^{-n}h \leq x$. Then according to $(*)$ and Lemma 1 we have $f(2^{-n}h) = 2^{-n}f(h)$. Therefore

$$\frac{f(h)}{h} = \frac{f(2^{-n}h)}{2^{-n}h} \leq \frac{f(x)}{x} \leq \frac{f(h)}{h},$$

which yields $f(x) = \dfrac{f(h)}{h} \cdot x$.

In contrast with Theorem 1 the following example shows that there is a continuous metric preserving function f such that each neighborhood of 0 contains an interval on which f is strictly convex.

Example 1. Define $f : [0, +\infty) \to [0, +\infty)$ as follows (see Fig. 8)

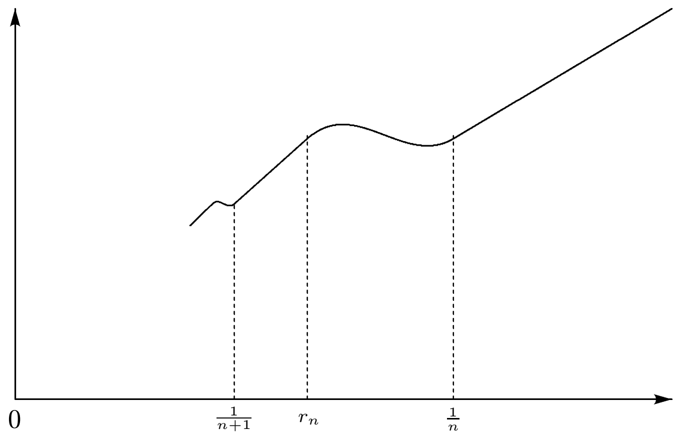


Figure 8

$$f(x) = \begin{cases} 0, & \text{if } x = 0, \\ \frac{2n+1}{n+1} \cdot x, & \text{if } x \in [\frac{1}{n+1}, r_n), \\ a_n x^3 + b_n x^2 + c_n x + d_n, & \text{if } x \in [r_n, \frac{1}{n}), \\ x, & \text{if } x \in [1, +\infty), \end{cases}$$

where

$$r_n = \frac{(2n-1)(n+1)}{(2n+1)\cdot n^2},$$

$$a_n = \frac{16n^7 + 24n^6 + 8n^5 - 2n^4 - n^3}{n+1},$$

$$b_n = \frac{-48n^6 - 72n^5 - 12n^4 + 18n^3 + 2n^2 - 2n}{n+1},$$

$$c_n = \frac{48n^6 + 72n^5 - 30n^3 + n^2 + 5n - 1}{n^2+n},$$

$$d_n = \frac{-16n^4 - 8n^3 + 12n^2 + 2n - 2}{n}.$$

5. An application of metric preserving functions

An interesting application of metric preserving functions was discovered by M. Jůza in 1956, long before the subject had matured [32]. It is now well known that there are complete nowhere discrete metric spaces that have a nested sequence of closed balls with empty intersection (but recall that the diameters of such balls cannot tend to 0). Jůza observed that the real line could be topologized to obtain such a space, using a metric preserving function; in particular, he showed that $(\mathbb{R}, e_f)$ has the required property if e is the usual metric on $\mathbb{R}$, and f is the metric preserving function defined in the following example.

Example 1. Define $f : [0, +\infty) \to [0, +\infty)$ as follows (see Fig. 9)

$$f(x) = \begin{cases} x & \text{if } x \le 2, \\ 1 + \frac{1}{x-1} & \text{if } x > 2. \end{cases}$$

Proposition 1 shows that f is metric preserving.

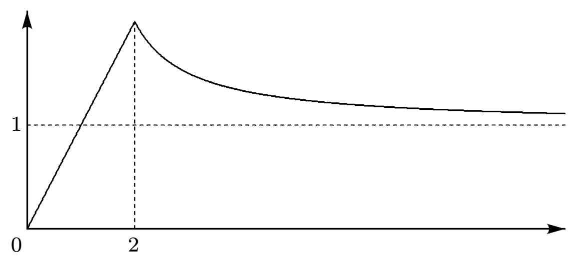


Figure 9

The following propositions enable to construct continuous metric preserving functions from tightly bounded functions.

Proposition 1. (See [4].) *For each function* $f : [0, +\infty) \to [0, +\infty)$ *and* $r > 0$ *define* $f_r : [0, +\infty) \to [0, +\infty)$ *as follows*

$$f_r(x) = \begin{cases} \dfrac{f(r)}{r} \cdot x & \textit{if } x \in [0, r), \\ f(x) & \textit{if } x \in [r, +\infty). \end{cases}$$

Let f *be metric preserving. Then* f_r *is metric preserving iff*

$$\forall x, y \in [r, +\infty) : \ |x - y| \le r \Rightarrow |f(x) - f(y)| \le \frac{f(r)}{r} \cdot |x - y|.$$

We will prove the following generalization of Proposition 1.

Proposition 2. (See [22].) *Let g, h be metric preserving. Let $r > 0$ be such that $g(r) = h(r)$. Define $f_{g,h,r} : [0, +\infty) \to [0, +\infty)$ as follows*

$$f_{g,h,r}(x) = \begin{cases} g(x), & \text{if } x \in [0, r), \\ h(x), & \text{if } x \in [r, \infty). \end{cases}$$

Suppose that g is nondecreasing and concave. Then $f_{g,h,r}$ is metric preserving iff

$$\forall x, y \in [r, \infty) : \ |x - y| \le r \Rightarrow |h(x) - h(y)| \le g(|x - y|).$$

Proof. One direction follows from Proposition 3.1. For the other direction, suppose $0 < a \le b \le c \le a + b$. We shall show that

$$(f_{g,h,r}(a), f_{g,h,r}(b), f_{g,h,r}(c)) \ \text{ is a triangle triplet.}$$

We distinguish two non-trivial cases.

a) Suppose that $a, b \in (0, r)$, and $c \in [r, +\infty)$. Evidently

$$f_{g,h,r}(a) \le f_{g,h,r}(b) \le f_{g,h,r}(b) + f_{g,h,r}(c).$$

Since $|g(r) - h(c)| \le g(|r - c|)$, we obtain

$$f_{g,h,r}(b) = g(b) \le g(r) + [g(a) - g(c - r)] \le g(a) + h(c) = f_{g,h,r}(a) + f_{g,h,r}(c).$$

Since g is concave, we have $g(r) + g(a + b - r) \le g(a) + g(b)$, which yields

$$f_{g,h,r}(c) \le g(r) + g(c - r) \le g(r) + g(a + b - r) \le f_{g,h,r}(a) + f_{g,h,r}(b).$$

b) Suppose that $a \in [0, r)$, and $b, c \in [r, +\infty)$. Since (r, b, c) is a triangle triplet, we obtain

$$f_{g,h,r}(a) \le g(r) = h(r) \le h(b) + h(c) = f_{g,h,r}(b) + f_{g,h,r}(c).$$

Since $|h(b) - h(c)| \le g(|b - c|)$, we have

$$f_{g,h,r}(b) = h(b) \le g(c - b) + h(c) \le g(a) + h(c) = f_{g,h,r}(a) + f_{g,h,r}(c),$$

and

$$f_{g,h,r}(c) = h(c) \le g(c - b) + h(b) \le g(a) + h(b) = f_{g,h,r}(a) + f_{g,h,r}(b).$$

Now, we begin with the following definition. Given a point x of a metric space (X, d) and a positive real number ε, the *closed ball* with the center x and radius ε is the set

$$B_d[x; \varepsilon] = \{y \in X;\ d(x, y) \leq \varepsilon\}.$$

It is well known that there is a complete metric space with the following property:

(1) *There is a monotone sequence of closed balls with empty intersection.*

In [32] it is shown that the metric space $(\mathbb{R}, e_f)$ has the property (1), where f is the function of Jůza (see Example 1). The proof of (1) is based on the following property of the metric space $(\mathbb{R}, e_f)$:

(2) *For each compact set K there is a closed ball $B_{e_f}[x; \varepsilon]$ and there is a compact set L such that $K \subset \mathbb{R} - B_{e_f}[x; \varepsilon] \subset L$.*

Theorem 1. (See [23].) *Let f be a metric preserving function. Suppose that there are functions $g, h : [0, +\infty) \to [0, +\infty)$ such that g and h are nonincreasing, and*
they are not constant in each neighborhood of the point $+\infty$, (3)
$g(x) \leq f(x) \leq h(x)$ in some neighborhood of the point $+\infty$, (4)
$\lim_{x \to +\infty} g(x) = \lim_{x \to +\infty} h(x)$. (5)
Then the metric space $(\mathbb{R}, e_f)$ has the property (2).

Proof. Let $m \in \mathbb{N}$ be such that $g(x) \leq f(x) \leq h(x)$ for each $x \in [m, +\infty)$. Put $d = \lim_{x \to +\infty} g(x)$. Evidently $d = \lim_{x \to +\infty} f(x) > 0$. Let K be a compact set. Put

$$s = \inf K - m, \quad r = \sup K - s, \quad \varepsilon = g(r).$$

Since g is not constant on $(r, +\infty)$, there is $\xi > r$ such that $g(\xi) \neq \varepsilon$.
Since g is nonincreasing, we have $\varepsilon \neq g(\xi) \leq g(r) = \varepsilon$. Therefore $g(\xi) < \varepsilon$.
Since g is nonincreasing for each $x \geq \xi$ we get $g(x) \leq g(\xi)$. Thus

$$d = \lim_{x \to +\infty} g(x) \leq g(\xi) < \varepsilon.$$

Let $x \in [m, r]$. Then $f(x) \geq g(x) \geq g(r) = \varepsilon$. Therefore

$$\forall x \in [m, r] : \; f(x) \geq \varepsilon. \tag{6}$$

Let $\delta \in (d, \varepsilon)$. Since $\lim_{x \to +\infty} h(x) = d < \varepsilon$, there is $t > r$ such that $h(t) < \delta$. Let $x \geq t$. Then $f(x) \leq h(x) \leq h(t) < \delta$. Thus

$$\forall x \in [t, +\infty) : \; f(x) < \delta. \tag{7}$$

Let S be a closed ball with the centre s and the radius δ. Put $L = [s-t, s+t]$. Now, we shall show that $K \subset \mathbb{R} - S$. Let $u \in K$. Then $|u-s| = u-s \in [m, r]$, and by (6) we get $e_f(u, s) = f(|u-s|) \geq \varepsilon > \delta$. Therefore $u \notin S$. Finally, we shall show that $\mathbb{R} - S \subset L$. Let $v \in \mathbb{R} - S$. Then $f(|v-s|) = e_f(v, s) > \delta$. By (7) we have $|v-s| < t$. Therefore $v \in L$.

Example 2. Define $f : [0, +\infty) \to [0, +\infty)$ as follows (see Figure 10):

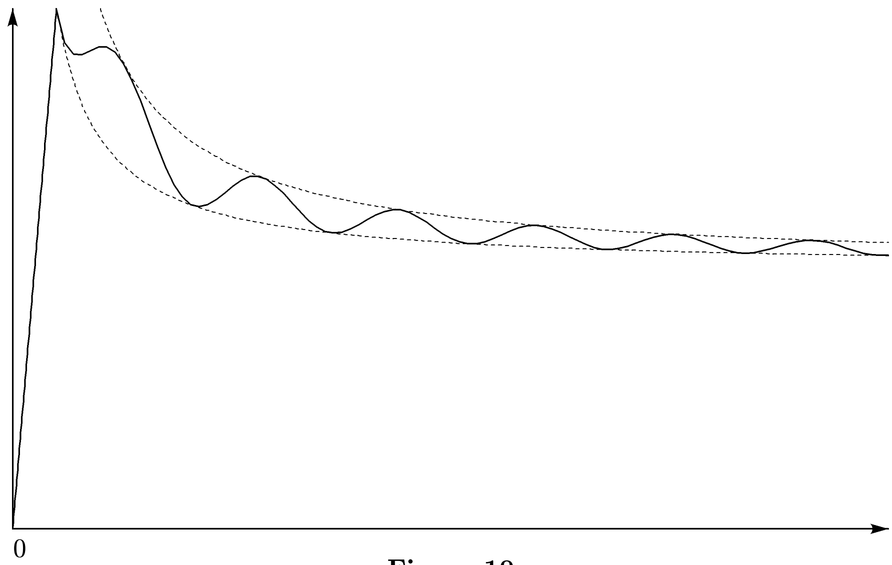


Figure 10

$$f(x) = \begin{cases} x & \text{if } x \in [0, 1), \\ \dfrac{1 + x + \sin^2(x-1)}{2x} & \text{if } x \in [1, +\infty). \end{cases}$$

It is not difficult to verify that f is metric preserving and the metric space $(\mathbb{R}, e_f)$ has the property (2) (which yields also the property (1)), however f is not monotone on every neighborhood of the point $+\infty$.

Example 3. Define $f : [0, +\infty) \to [0, +\infty)$ as follows (see Fig. 11):

$$f(x)=x \text{ if } x\in[0,1], \text{ and}$$
$$f(x)=\tfrac{1}{2}\cdot\Big(x-3n+1-|x-3n+1|+\Big|x-3n+\tfrac{1}{2}+\tfrac{1}{n+2}\Big|+\Big|x-3n-\tfrac{1}{2}-\tfrac{1}{n+2}\Big|\Big),$$
$$\text{if } x\in(3n-2,3n+1] \quad (n\in\mathbb{N}).$$

It is not difficult to verify that f is metric preserving and $(\mathbb{R}, e_f)$ is a metric space with the property (1), which has not the property (2).
Indeed, the intersection of the sequence of closed balls $B_{e_f}[x_n; \varepsilon_n]$ with the center $x_n = 3 \cdot (2^{n-1} - 1)$ and the radius $\varepsilon_n = \frac{1}{2} + \frac{1}{2^n+1}$ is empty.

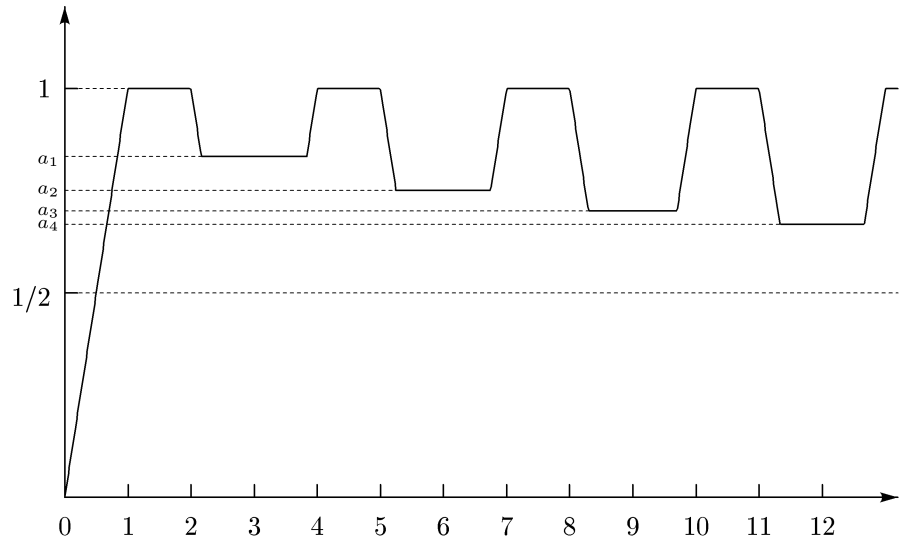


Figure 11

A characterization of metric preserving functions f such that the space $(\mathbb{R}, e_f)$ has the property (1) remains an open question.

6. Metric preserving functions and periodicity

The examples given in the previous chapter show, in particular, that metric preserving functions need not be nondecreasing. I. Pokorný [49] has isolated a fairly natural class of amenable functions for which all metric preserving functions must be nondecreasing. Define

$$\begin{aligned}\mathcal{G} = \{ & f \in \mathcal{O};\ \text{there is a periodic function } g \\ & \text{such that } \ \forall x \geq 0 : \ f(x) = x + g(x)\}.\end{aligned}$$

Examples of such metric preserving functions are $x \mapsto x + |\sin(x)|$ (see Fig. 12) and $x \mapsto \lfloor x \rfloor + \sqrt{x - \lfloor x \rfloor}$ (see Fig. 13); where $x \mapsto \lfloor x \rfloor$ is the *floor function*, which returns the largest integer not greater than its argument.

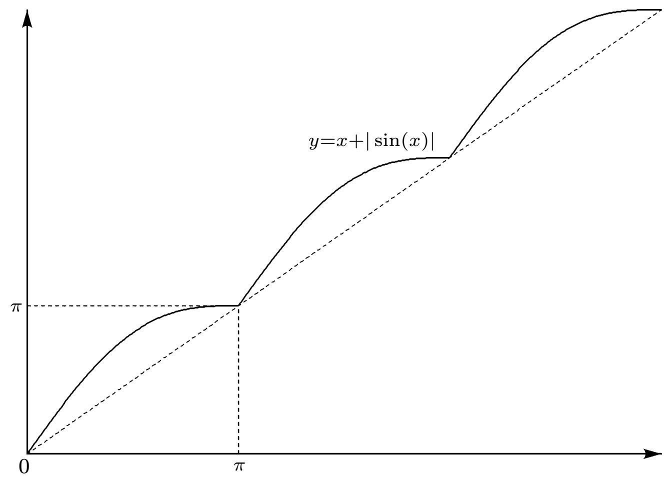


Figure 12

We denote by ι the identity function on $[0, +\infty)$ (i.e. $\iota(x) = x$ for each $x \geq 0$). The previous two metric preserving functions have the following property

(∗) $\qquad f - \iota$ *is periodic and nonconstant.*

It is easy to see that $f : [0, +\infty) \to [0, +\infty)$ is subadditive iff $f - \iota$ is subadditive.

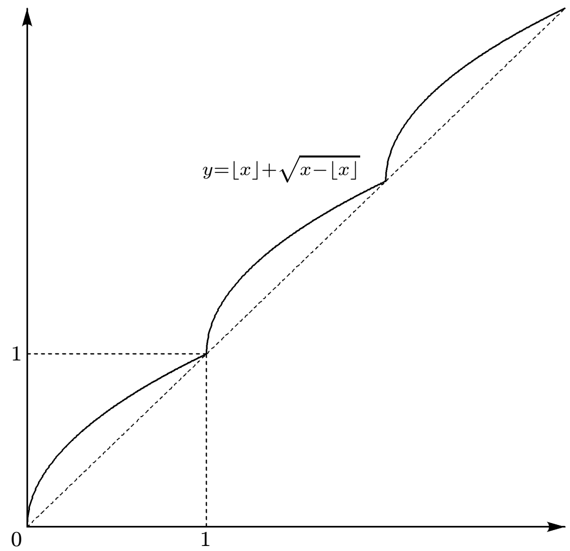


Figure 13

Lemma 1. (See [7].) *Let f be metric preserving, $k > 0$. If in each neighborhood of 0 there is a point a such that $f(a) = ka$, then $f(x) = kx$ holds in a suitable neighborhood of 0.*

Proof. Let $h > 0$ be such that $f(h) = kh$. We shall show that $f(x) = kx$ for each $x \in [0, h]$. Assume that $f(x) \neq kx$ for some $x \in (0, h)$. We distinguish two cases.

1) Suppose that $f(x) > kx$. Put

$$A = \{y \in [0, +\infty) : f(y) = ky\}.$$

Since f is continuous (by Theorem 3.1), the set $A \cap [0, x]$ is closed and bounded. Put $M = \max(A \cap [0, x])$. Let $y \in A$ be such that $0 < y < x - M$. Then

$$f(M + y) \leq f(M) + f(y) = kM + ky = k(M + y).$$

Since $f(x) > kx$ and since f is continuous, there is $z \in [M + y, x]$ such that $f(z) = kz$, which contradicts the definition of M.

2) Suppose that $f(x) < kx$. Evidently the set $A \cap [x, h]$ is closed and bounded. Put $m = \min(A \cap [x, h])$. Let $r \in A$ be such that $0 < r < m - x$. Then

$$km = f(m) \leq f(m - r) + f(r) = f(m - r) + kr,$$

which yields $f(m - r) \geq km - kr = k(m - r)$. Since $f(x) < kx$ and f is continuous, there is $s \in [z, m - r]$ such that $f(s) = ks$, which contradicts the definition of m.

Lemma 2. *Let $f \in \mathcal{G}$ be metric preserving, $f \neq \iota$. Then $f - \iota$ has the smallest period.*

Proof. Put $g = f - \iota$. Suppose there does not exist the smallest period of g. By Lemma 1 there is a neighbourhood U of 0 on which $f(x) = x$ and hence $g(x) = 0$ on U. Then from periodicity of g it follows that $g \equiv 0$.

Proposition 1. *Let $f \in \mathcal{G}$ be metric preserving. Then f is nondecreasing.*

Proof. Put $g = f - \iota$. Denote by p the smallest period of g. First we show that f is nondecreasing on $(0, p)$. We prove it by contradiction. Suppose that there are $x_1, x_2 \in (0, p)$ such that $x_1 < x_2$ and $f(x_1) > f(x_2)$. Let $a = x_1 + p$, $b = p$ and $c = x_2$. Then (a, b, c) is a triangle triplet and by Theorem 2.1

$$\begin{aligned} f(a) &\leq f(b) + f(c) = f(p) + f(x_2) = p + f(x_2) < \\ &< p + f(x_1) = p + x_1 + g(x_1) = \\ &= x_1 + p + g(x_1 + p) = f(x_1 + p) = f(a), \end{aligned}$$

a contradiction.

Since for each $k \in \mathbb{N}$ and $x \in (0, p)$ we have

$$f(x + kp) = x + kp + g(x + kp) = x + kp + g(x) = f(x) + kp,$$

the function f is nondecreasing on $[0, +\infty)$.

As an immediate corollary we obtain

Theorem 1. (See [49].) *$f \in \mathcal{G}$ is metric preserving iff f is nondecreasing and subadditive.*

Lemma 3. *Let $f \in \mathcal{G}$ be metric preserving, $f \neq \iota$. Then $f(x) \geq x$ for all $x \in [0,p)$, where p is the smallest period of $f - \iota$.*

Proof. By contradiction. Suppose that there is $a \in (0,p)$ such that $f(a) < a$. Then $a - f(a) > 0$ and there is $k \in \mathbb{N}$ such that

$$k \cdot (a - f(a)) > p. \tag{1}$$

Let $\ell \in \mathbb{N}$ be such that $\ell \cdot p \leq k \cdot a < (\ell + 1) \cdot p$. Then

$$0 \leq ka - p\ell < p. \tag{2}$$

Put $g = f - \iota$. Acording to subadditivity of g and the inequalities (2), (1) we have:

$$\begin{aligned} f(ka - p\ell) &= (ka - p\ell) + g(ka - p\ell) = \\ &= (ka - p\ell) + g(ka) \leq \\ &\leq (ka - p\ell) + k \cdot g(a) = \\ &= (ka - p\ell) + k \cdot (f(a) - a) < p - p = 0, \end{aligned}$$

i.e., $f(ka - p\ell) < 0$, a contradiction.

Lemma 4. *Let $f \in \mathcal{G}$ be metric preserving, $f \neq \iota$. Put $g = f - \iota$. Suppose that there are relatively prime positive integers m, n such that $g\left(\frac{m}{n} \cdot p\right) = 0$, where p is the smallest period of g. Then for each $i \in \mathbb{N}$ we have*

$$g\left(\frac{i}{n} \cdot p\right) = 0.$$

Proof. Let $k, \ell \in \mathbb{N}$ such that $k \cdot m = \ell \cdot n + 1$. Then by subadditivity of g we have

$$0 = g\left(k \cdot \frac{m}{n} \cdot p\right) = g\left(\frac{\ell \cdot n + 1}{n} \cdot p\right) = g\left(\ell \cdot p + \frac{p}{n}\right) = g\left(\frac{p}{n}\right).$$

By subadditivity of g we obtain $g\left(i \cdot \frac{p}{n}\right) = 0$ for every $i \in \mathbb{N}$.

Theorem 2. (See [49].) *Let $f \in \mathcal{G}$ be metric preserving, $f \neq \iota$. Put $g = f - \iota$. Then $g(x) > 0$ for every $x \in (0, p)$, where p is the smallest period of g.*

Proof. By contradiction. Suppose that there is $a \in (0, p)$ such that $g(a) = 0$.

1) Suppose that there are relatively prime positive integers m, n such that $a = \frac{m}{n} \cdot p$. By Lemma 4 we have $g\left(\frac{p}{n}\right) = 0$. Let $x \in (0, \frac{p}{n})$ and let $k \in \mathbb{N} \cap (1, n)$. Then from subadditivity of g we obtain

$$g(x) = g(x) + g\left(k \cdot \frac{p}{n}\right) \geq g\left(x + k \cdot \frac{p}{n}\right) = g\left(x + k \cdot \frac{p}{n}\right) + \\ + g\left((n-k) \cdot \frac{p}{n}\right) \geq g\left(x + k \cdot \frac{p}{n} + (n-k) \cdot \frac{p}{n}\right) = \\ = g(x+p) = g(x).$$

Therefore $g\left(x + k \cdot \frac{p}{n}\right) = g(x)$ which shows that $\frac{p}{n}$ is a period of g. This contradicts the definition of p.

2) Suppose that $\dfrac{a}{p}$ is irrational. It is well-known that for arbitrary irrational number x the set $\{k \cdot x - \lfloor k \cdot x \rfloor;\ k \in \mathbb{N}\}$ is dense in $[0, 1]$. Put

$$A = \left\{k \cdot \tfrac{a}{p} - \left\lfloor k \cdot \tfrac{a}{p} \right\rfloor;\ k \in \mathbb{N}\right\}.$$

Then the set $B = p \cdot A = \{p \cdot x;\ x \in A\}$ is dense in $[0, p]$. Therefore $g(x) = 0$ for every $x \in B$ (since $x = k \cdot a - \ell \cdot p$ for suitable $k, \ell \in \mathbb{N}$). By Lemma 1 there is a neighbourhood U of 0 such that $f \equiv \iota$ on U, i.e. $g \equiv 0$ on U. Choose relatively prime positive integers m, n such that $\frac{m}{n} \cdot p \in (0, p) \cap U$ (which evidently yields $g(\frac{m}{n} \cdot p) = 0$). But such case was discussed in the previous part of this proof.

7. Metric preserving functions and differentiability

Mirroring the situation for continuity, the notion of differentiability partitions the class of metric preserving functions into two rather different subclasses, and the assignment of a given metric preserving function to one of these subclasses is determined by the value of its derivative at 0 (of course, there is a one-sided derivative at 0 only). As we shall see, for such functions, the (extended) derivative always exists at 0; the central question becomes whether the derivative is finite or infinite. Those with finite derivative form a well-behaved class of continuous functions that are differentiable almost everywhere; those with infinite derivative, by contrast, can be very unruly: they can be continuous, nowhere differentiable (in the finite sense), and even, as we saw in Chapter 2, nowhere continuous.

Lemma 1. (See [60].) *Let $f \in \mathcal{O}$ be a differentiable function. If f is metric preserving, then the following conditions are fulfilled:*

(1) $f'(x) \leq f'(0)$ *for all* $x \in [0, +\infty)$.
(2) $f'(0) > 0$.

Proof. Fix $x \in [0, +\infty)$. Subadditivity ensures that for all $h > 0$,

$$\frac{f(x+h) - f(x)}{h} \leq \frac{f(h) - f(0)}{h}.$$

For $h \to 0$ we get $f'(x) \leq f'(0)$.

Assume now $f'(0) \leq 0$. Then the already proved condition (1) implies $f'(x) \leq 0$ for all $x \in [0, +\infty)$, thus f is decreasing, hence $f(x) \leq f(0) = 0$, a contradiction.

Proposition 1. (See [60].) *Let f be metric preserving. If f is differentiable over $(u, +\infty)$ for some $u \geq 0$ and $\lim_{x \to +\infty} f'(x) = +\infty$, then f is not metric preserving.*

Proof. Fix $m > u$. Subadditivity ensures that for all $x > m$,

$$\frac{f(x+m) - f(x)}{m} \leq \frac{f(m)}{m}.$$

Now use the Mean Value Theorem to show that for arbitrary large x, there exists $x_0 \in (x, x+m)$ such that $f'(x_0) \leq \frac{f(m)}{m}$.

The following example shows that the assumption

$$\lim_{x\to+\infty} f'(x) = +\infty$$

in Proposition 1 cannot be replaced by

$$\limsup_{x\to+\infty} f'(x) = +\infty.$$

Example 1. (See [7].) There is a metric preserving function f such that (see Fig. 14)

(3) f is continuous,
(4) f is differentiable on $(0, +\infty)$,
(5) $\limsup\limits_{x\to+\infty} f'(x) = +\infty$.

Put $f = \sum\limits_{n=1}^{\infty} f_n$, where $a_n = 1 - \sqrt{1 - 2^{-2n}}$, and $f_n : [0, +\infty) \to [0, +\infty)$,

$$f_n(x) = \begin{cases} (2^n a_n)^{-1} \cdot \sqrt{2a_n x - x^2}, & \text{if } x \in [0, a_n], \\ 2^{-n-1}\left[3 + \cos\left(\frac{2(x-n-1)}{a_n}\right)\right], & \text{if } x \in [n+1-\frac{\pi}{2}a_n, n+1+\frac{\pi}{2}a_n], \\ 2^{-n}, & \text{otherwise.} \end{cases}$$

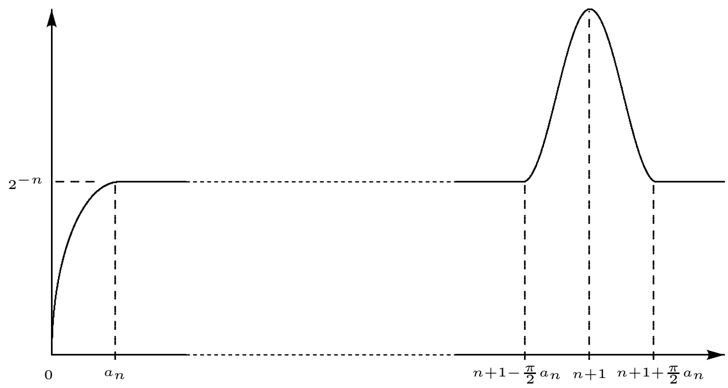


Figure 14

Proposition 2. (See [60].) *Let $f \in \mathcal{O}$ be differentiable and let f' be continuous on a certain interval $[0, u)$, $u > 0$. If f is metric preserving, then it is increasing on some neighborhood of 0.*

Proof. By Lemma 1 we have $f'(0) > 0$. Since f' is continuous on $[0, u)$, there is $v > 0$ such that $f'(x) > 0$ on $[0, v)$, hence f is increasing on $[0, v)$.

Example 4.1 shows that there is a metric preserving function f such that f is differentiable, f' is continuous on $(0, +\infty)$, and f is not increasing on each neighborhood of 0. Therefore this example shows that the assumption

$$f \textit{ is continuous on a certain interval } [0, u), u > 0$$

in Proposition 2 is essential.

Now we show that for each metric preserving function f, $f'(0)$ exists in the extended sense. The proof naturally divides into two parts depending on whether the following set is empty:

$$K_f = \{k > 0 : \ f(x) \leq kx \text{ for all } x \geq 0\}.$$

In the course of the proof, we show that $f'(0) < +\infty$ iff $K_f \neq \emptyset$, and $f'(0) = +\infty$ iff $K_f = \emptyset$.

Lemma 2. (See [7].) *If f is metric preserving, then for all $x, y > 0$ we have*

$$x \geq y \Rightarrow \frac{f(x)}{x} \leq 2 \cdot \frac{f(y)}{y}.$$

Proof. Let $0 < x \leq y$. Choose a positive integer n such that

$$2^{n-1} \leq xy^{-1} < 2^n.$$

Since $2^{1-n}x < 2y$, by Corollary 2.1 $f(2^{1-n}x) \leq 2f(y)$. By Lemma 4.1 we have $2^{1-n}f(x) \leq f(2^{1-n}x)$. Thus

$$f(x) \leq 2^{n-1}f(2^{1-n}x) \leq 2^{n-1}2f(y) \leq xy^{-1}2f(y).$$

We can now prove that $f'(0)$ exists and is infinite when $K_f = \emptyset$. Let n be a positive integer. Since $K_f = \emptyset$, we can pick $y > 0$ such that $f(y) \geq 2ny$.

Let $x \in (0, y]$. By Lemma 2, $2n \leq \frac{f(y)}{y} \leq 2\frac{f(x)}{x}$. But now we have shown that for each integer $n > 0$, there is $y > 0$ such that if $0 < x \leq y$, $\frac{f(x)}{x} \geq n$, as required.

We turn to the case in which $K_f \neq \emptyset$. We prove that $f'(0)$ exists and is finite in this case. By Theorem 3.1, f is continuous, whence K_f is closed. Let $k_0 = \min K_f$. We show

$$k_0 = \lim_{x \to 0} \frac{f(x)}{x}. \tag{6}$$

Let $\varepsilon > 0$. Then, by the choice of k_0,

$$\forall h > 0 \; \exists x \in (0, h] : \; f(x) > (k_0 - \varepsilon)x. \tag{7}$$

We show that

$$\exists h > 0 \; \forall x \in (0, h] : \; f(x) > (k_0 - \varepsilon)x. \tag{8}$$

Assume instead that

$$\forall h > 0 \; \exists x \in (0, h] : \; f(x) \leq (k_0 - \varepsilon)x. \tag{9}$$

Let $h > 0$. By the formula (7), there is $x_1 \in (0, h]$ such that

$$f(x_1) > (k_0 - \varepsilon)x_1,$$

and by (9), there is $x_2 \in (0, h]$ such that $f(x_2) \leq (k_0 - \varepsilon)x_2$. By the continuity of f, there is $x_3 \in (0, h]$ such that $f(x_3) = (k_0 - \varepsilon)x_3$. By Lemma 6.1, $f(x) = (k_0 - \varepsilon)x$ holds on some neighborhood of 0, contradicting (7). This proves (8).

Now since $k_0 \in K_f$, we also have $f(x) < (k_0 + \varepsilon)x$ for each $x > 0$. Thus, combining these results, we obtain

$$\forall \varepsilon > 0 \; \exists h > 0 \; \forall x \in (0, h] : \; k_0 - \varepsilon < \frac{f(x)}{x} < k_0 + \varepsilon;$$

that is, (6) holds, as required.

We have proved the following

Theorem 1. (See [7].) *For every metric preserving function f, $f'(0)$ always exists (in the extended sense) and $f'(0) = \inf K_f$. (We put $\inf \emptyset = +\infty$.)*

Given $k > 0$, we say that a function $f \in \mathcal{O}$ is of *k-bounded gradient at* 0 if there is $h > 0$ such for all $x \in [0, h]$, $f(x) \leq kx$. We say that f is of *bounded gradient at* 0 if for some $k > 0$, f is of k-bounded gradient at 0. (See [62].)

Lemma 3. (See [7].) *Suppose $k > 0$ and f is metric preserving and of k-bounded gradient at 0. Then*

(10) $\forall x \in [0, +\infty) : f(x) \leq kx$, *and*
(11) $\forall x, y \in [0, +\infty) : |f(x) - f(y)| \leq k \cdot |x - y|$.

Proof. Let $x \in [0, +\infty)$. Let n be a positive integer such that $2^{-n}x \leq h$. By Lemma 4.1 $2^{-n}f(x) \leq f(2^{-n}x) \leq k \cdot 2^{-n}x$, which yields (10). Observe that (11) follows from Proposition 3.1 and (10).

If $f'(0) < +\infty$, then f is a Lipschitz mapping with the Lipschitz constant $f'(0)$ (which yields that f is differentiable almost everywhere), as the following theorem shows.

Theorem 2. *Let f be a metric preserving function with $f'(0) < +\infty$. Then*

(12) $\forall x \in [0, +\infty) : f(x) \leq f'(0) \cdot x$, *and*
(13) $\forall x, y \in [0, +\infty) : |f(x) - f(y)| \leq f'(0) \cdot |x - y|$.

Proof. Let $\varepsilon > 0$. Then there is $h > 0$ such that $f(x) \leq (f'(0) + \varepsilon) \cdot x$ for each $x \in [0, h]$. By Lemma 3 $f(x) \leq (f'(0) + \varepsilon) \cdot x$ for each $x \in [0, +\infty)$. Since $\varepsilon > 0$ was arbitrary, (12) holds. Observe that (13) follows from Proposition 3.1 and (12).

Corollary 1. *Suppose f is metric preserving and $f'(0) < +\infty$. If the extended derivative of f exists at each $x \in [0, +\infty)$, then $|f'(x)| \leq f'(0)$ for each $x \in [0, +\infty)$.*

The proof of Theorem 2 shows that if f is a metric preserving function and $f'(0) < +\infty$, then f is of bounded gradient at 0. The converse is also true, and follows immediately from Theorem 1. Thus:

Corollary 2. *For metric preserving functions f, $f'(0) < +\infty$ iff f is of bounded gradient at 0.*

Our target theorem follows directly from Theorem 2.

Theorem 3. *Suppose f is metric preserving and $f'(0) < +\infty$. Then f is of bounded variation over each closed interval lying in $[0, +\infty)$.*

Finally, we consider the subclass of metric preserving functions f such that $f'(0) = +\infty$. As we mentioned before, the most pathological examples are possible in this case. In contrast with Theorem 2 we will construct a continuous metric preserving function which is nowhere differentiable. This function is a slight modification of the Van der Waerden's continuous nowhere differentiable function. (See [3].)

Example 2. (See [22].) Define $h : [0, +\infty) \to [0, +\infty)$ as follows (see Fig. 15)

$$h(x) = \begin{cases} x, & \text{if } x \leq \frac{1}{2}, \\ \frac{1}{2} + |x - \lfloor x \rfloor - \frac{1}{2}|, & \text{if } x > \frac{1}{2}. \end{cases}$$

Define $f : [0, +\infty) \to [0, +\infty)$ as follows

$$f(x) = \sum_{n=0}^{\infty} 2^{-n} h(2^n x) \text{ for each } x \in [0, +\infty).$$

The proof that f is continuous and nowhere differentiable is essentially the same as Van der Waerden's. It is not difficult to verify that f is metric preserving.

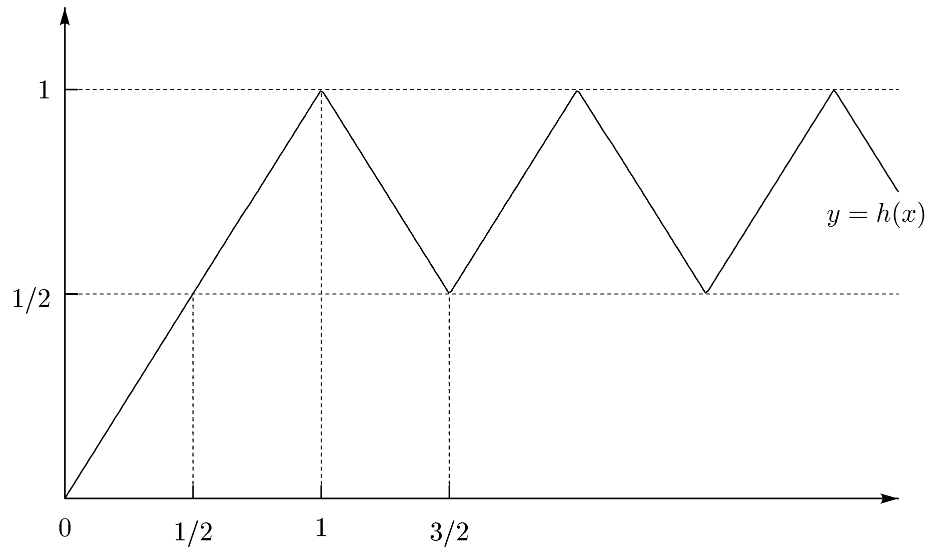


Figure 15

We close this section by considering a question raised by the authors in [22] regarding metric preserving functions:

It is possible to characterize the set $f'^{-1}(+\infty)$ *?*

By Corollary 1 the question is relevant only to the case we are currently considering, where $f'(0) = +\infty$.

The following example shows that there is a monotone continuous metric preserving function f for which

$$f'^{-1}(+\infty) = \{0\} \cup \{2^{-n} : n \in \mathbb{N}\}.$$

Example 3. (See [22].) There is a metric preserving function f such that

(14) f is continuous and nondecreasing,
(15) $f'(x)$ exists for each $x \in [0, +\infty)$ (finite or infinite),
(16) $f'(2^{-n}) = +\infty$ for each $n \in \mathbb{N}$.

Define $g : [0, +\infty) \to [0, +\infty)$ as follows

$$g(x) = \begin{cases} \sqrt{2x - x^2}, & \text{if } x \in [0, 1), \\ 1, & \text{if } x \in [1, +\infty). \end{cases}$$

Evidently g is nondecreasing and concave. Define $h : [0, +\infty) \to [0, +\infty)$ as follows

$$h(x) = \begin{cases} 0, & \text{if } x = 0, \\ 1, & \text{if } x \in (0, 1), \\ \frac{1}{2} \cdot [3 - g(2 - x)], & \text{if } x \in [1, 2), \\ \frac{1}{2} \cdot [3 + g(x - 2)], & \text{if } x \in [2, +\infty). \end{cases}$$

Since for all $x > 0$ we have $1 \leq h(x) \leq 2$, by Proposition 2.4 h is metric preserving. We shall show that the assumptions of Propositon 5.2 are fulfiled.

Let $x, y \in [1, +\infty)$, $|x - y| \leq 1$. We distinguish three cases.

a) Suppose that $1 \leq x \leq y < 2$. Since $2 - x = (2 - y) + (y - x)$, we have $g(2 - x) \leq g(2 - y) + g(y - x)$. Thus $|h(x) - h(y)| = \frac{1}{2} \cdot [g(2 - x) - g(2 - y)] \leq \frac{1}{2} \cdot g(y - x) \leq g(|x - y|)$.

b) Suppose that $1 \leq x < 2 \leq y$. Since g is nondecreasing, we obtain $g(2-x) \leq g(y-x)$ and $g(y-2) \leq g(y-x)$. Therefore $|h(x)-h(y)| = \frac{1}{2} \cdot [g(2 - x) + g(y - 2)] \leq \frac{1}{2} \cdot [g(y - x) + g(y - x)] = g(|x - y|)$.

c) Suppose that $2 \leq x \leq y$. Since $y - 2 = (y - x) + (x - 2)$, we have $g(y - 2) \leq g(y - x) + g(x - 2)$. Thus $|h(x) - h(y)| = \frac{1}{2} \cdot [g(y - 2) - g(x - 2)] \leq \frac{1}{2} \cdot g(y - x) \leq g(|x - y|)$.

Define $w : [0, +\infty) \to [0, +\infty)$ as follows (see Fig. 16)

$$w(x) = \begin{cases} g(x), & \text{if } x \in [0, 1), \\ h(x), & \text{if } x \in [1, +\infty). \end{cases}$$

By Proposition 5.2 w is metric preserving. It is not difficult to verify that

- w is continuous and nondecreasing,
- $w(x) \leq 2$ for each $x \in [0, +\infty)$,
- $w(x) = 2$ for each $x \geq 3$,
- $w'(x)$ exists for each $x \in [0, +\infty)$ (finite or infinite),
- $w'(2) = +\infty$.

Define $f : [0, +\infty) \to [0, +\infty)$ as

$$f(x) = \sum_{n=0}^{\infty} 2^{-n} w(2^n x) \text{ for each } x \in [0, +\infty).$$

It is not difficult to verify that (14)–(16) hold.

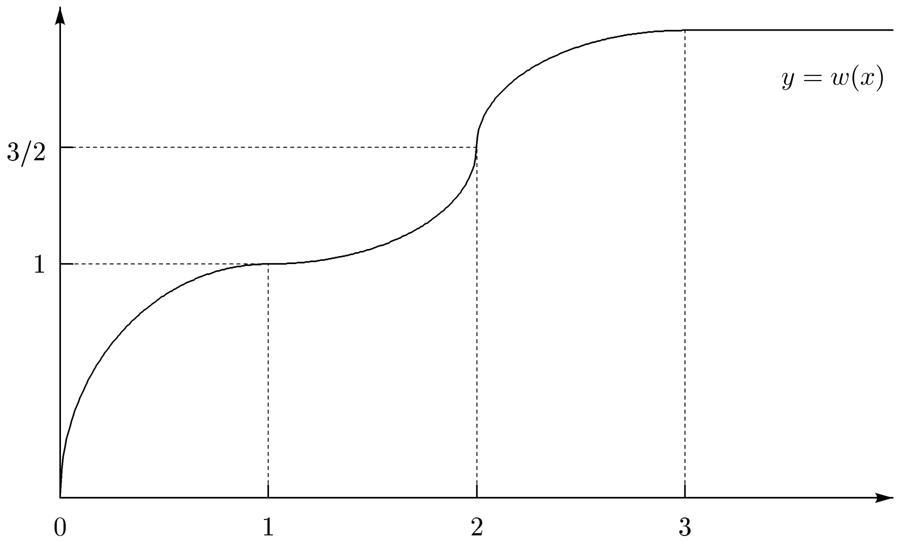


Figure 16

R. W. Vallin [65] generalizes this considerably by showing that for each G_δ measure zero set Z there is a continuous metric preserving function f such that

$$f'^{-1}(+\infty) = \{0\} \cup Z.$$

Vallin'sargument is technically interesting. As a starting point, he uses the following result mentioned in [10]:

Lemma 4. *Suppose $Z \subset [0,1]$ is a G_δ set of measure zero. Then there is an absolutely continuous function g defined on $[0,1]$ such that $g'^{-1}(+\infty) = Z$ and $g'(x) \geq 1$ for all $x \in [0,1] - Z$.*

Vallin's task is to modify the g of Lemma 4 so that it retains the properties in the lemma but becomes metric preserving. First he replaced g by f defined by

$$f(x) = \begin{cases} 0, & \text{if } x = 0, \\ \frac{2}{\pi} \cdot \arctan(g(x)) + 1, & \text{if } x \in (0,1]. \end{cases}$$

Then

1. f is metric preserving,
2. f' exists (in the extended sense) at all x, and
3. $f'^{-1}(+\infty) = \{0\} \cup Z$.

This function is, however, not continuous at the origin. Vallin's next goal is to construct a metric preserving function with infinite derivative on $\{0\} \cup Z$ which is continuous on all of $[0, 1]$. To build the required function, he makes use of the technique described in Proposition 5.2.

Let g be the function from Lemma 4 which is absolutely continuous on $[0, 1]$ and

$$g'(x) = +\infty \text{ for } x \in Z, \text{ while}$$
$$g'(x) \text{ exists and is finite for } x \notin Z.$$

Define $\hat{g}(x) = \frac{2}{\pi} \arctan(g(x)) + 1$. On $[0, 1]$, this $\hat{g}$ is not just continuous, but uniformly continuous. So we can rate at which $\hat{g}'(x)$ becomes infinite on $\{0\} \cup Z$. Since $\hat{g}$ is uniformly continuous, for each $n \in \mathbb{N}$ there exists a $\delta_n > 0$ such that for all $x \in [0, 1]$

$$\text{if } |x - y| < \delta_n \text{ then } |\hat{g}(x) - \hat{g}(y)| \leq 2^{-n}.$$

Now for small h values there is an n such that $h \in [\delta_{n+1}, \delta_n)$ and so

$$\tfrac{1}{h} \cdot [\, \hat{g}(x + h) - \hat{g}(x)] \leq 1/(2^n \delta_{n+1})$$

for each $x \in [0, 1]$.

Let $s : [0, 1] \to [0, 1]$ be an increasing, differentiable, concave function such that $s(0) = 0$ and for all n

$$\frac{s(\delta_n)}{\delta_n} \geq 1/(2^n \delta_{n+1}).$$

Using this s we can construct a sequence of continuous, differentiable metric preserving functions f_n.

Start with $(a_n)_{n \in \mathbb{N}}$, a sequence of points not in Z converging to zero. For each n find the point b_n such that $s(b_n) = \hat{g}(a_n)$. If $b_n > \frac{1}{2} a_n$ define

$$f_n(x) = \begin{cases} s(2b_n x / a_n) & \text{on } [0, a_n/2], \\ t(x) & \text{on } [a_n/2, a_n], \\ \hat{g}(x) & \text{on } [a_n, 1], \end{cases}$$

where t is a differentiable spline with range $[1,2]$ satisfying

$$|t(x)-t(y)| \leq s\left(\frac{2b_n}{a_n}|x-y|\right) \text{ for } |x-y| \leq a_n/2.$$

If $b_n \leq a_n/2$ let

$$f_n(x) = \begin{cases} s(x) & \text{on } [0,b_n], \\ t(x) & \text{on } [b_n,a_n], \\ \hat{g}(x) & \text{on } [a_n,1], \end{cases}$$

where again t is a differentiable spline with range $[1,2]$ now satisfying

$$|t(x)-t(y)| \leq s(|x-y|) \text{ for } |x-y| \leq b_n.$$

From Proposition 5.2 each f_n is metric preserving and $f_n'(x) = +\infty$ on $\{0\} \cup ([a_n,1] \cap Z)$. Last, define

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} f_n(x).$$

Vallin's function f has domain $[0,1]$; to be truly metric preserving, the domain needs to be $[0,+\infty)$. This is easily accomplished by replacing f by $f(h)$ where $h(x) = \frac{x}{1+x}$, restricted to $[0,+\infty)$. (See Corollary 0.2.)

8. The standard Cantor function is metric preserving

The usual definition of the standard Cantor function ("the devil's staircase") involves the classic middle-thirds description of the standard Cantor set. (See [16] and [52].) We offer an alternate definition of this function.

Define a sequence of functions $\phi_n : \mathbb{R} \longrightarrow [0,1]$ by

$$\phi_0(x) = \begin{cases} 0 & \text{if } x \le 0 \\ x & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x \ge 1 \end{cases} \quad \phi_{n+1}(x) = \begin{cases} \frac{1}{2} \cdot \phi_n(3x) & \text{if } x \le \frac{2}{3} \\ \frac{1}{2} + \frac{1}{2} \cdot \phi_n(3x-2) & \text{if } x \ge \frac{1}{3} \end{cases}$$

It is easy to check that each ϕ_n is non-decreasing, that $\phi_n(x) = 0$ for all $x \le 0$, that $\phi_n(x) = 1$ for all $x \ge 1$, and that the two lines in the definition of ϕ_{n+1} agree in the overlap of their domains, both giving $\phi_{n+1}(x) = \frac{1}{2}$ when $\frac{1}{3} \le x \le \frac{2}{3}$.

Put $\phi = \lim_{n\to+\infty} \phi_n$.

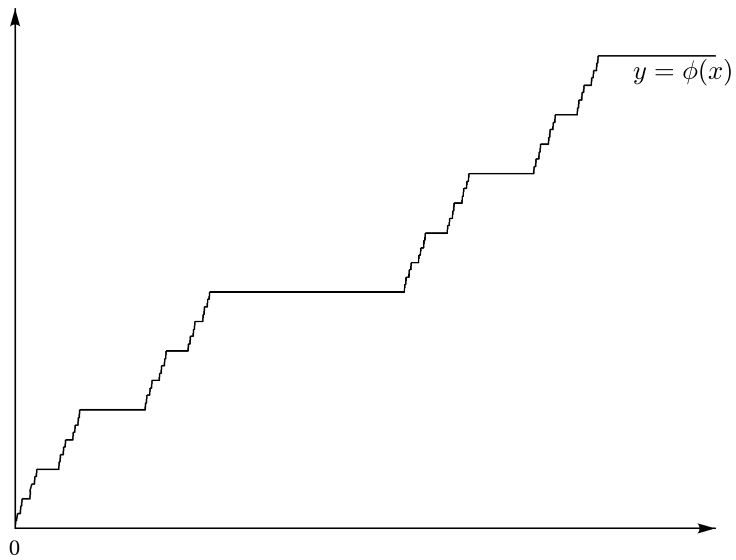


Figure 17

It is not difficult to verify that the restriction of ϕ to $[0,1]$ is the standard Cantor function φ. (See Fig. 17.) The functions ϕ_n are polygonal approximations of ϕ.

W. Sierpiński in 1911 gave the following characterization of the standard Cantor function (using a system of three functional equations).

Proposition 1. (See [57].) *There is a unique function* $\varphi : [0,1] \to [0,1]$ *which satisfies for each* $t \in [0,1]$ *the equations*

$$\varphi\left(\tfrac{t}{3}\right) = \tfrac{\varphi(t)}{2}, \quad \varphi\left(\tfrac{t+1}{3}\right) = \tfrac{1}{2}, \quad \varphi\left(\tfrac{t+2}{3}\right) = \tfrac{1}{2} + \tfrac{\varphi(t)}{2}.$$

This function is continuous, increasing, and possesses a dense set of intervals of constancy.

Proposition 2. (See [13].) *There is a unique function* $\varphi : [0,1] \to [0,1]$ *satisfying the conditions:*

$$\begin{aligned} \varphi(x) &= 1/2 && \textit{for } x \in [\tfrac{1}{3}, \tfrac{2}{3}], \\ \varphi(x) &= 2\varphi\left(\tfrac{x}{3}\right) && \textit{for } x \in [0,1], \\ \varphi(x) &= 1 - \varphi(1-x) && \textit{for } x \in [0,1]. \end{aligned}$$

Theorem 1. *The standard Cantor function is subadditive.*

Proof. The function ϕ is the pointwise limit of the functions ϕ_n as $n \to +\infty$. So to prove the subadditivity of ϕ, it suffices to prove the subadditivity of all ϕ_n, which we do by induction on n. The case $n = 0$ is trivial, so we proceed to the induction step from n to $n+1$. Let $x, y \in \mathbb{R}$, $x \geq y$. Here we consider several cases.

Case 1: $y \leq 0$. This case is trivial as f_{n+1} is monotone.

Case 2: $y \geq \frac{1}{3}$. In this case,

$$\phi_{n+1}(x+y) \leq 1 = \tfrac{1}{2} + \tfrac{1}{2} \leq \phi_{n+1}(x) + \phi_{n+1}(y).$$

Case 3: $x \leq \frac{1}{3}$. As x, y and $x+y$ are all $\leq \frac{2}{3}$, we have

$$\begin{aligned} \phi_{n+1}(x+y) &= \tfrac{1}{2} \cdot \phi_n(3x+3y) \leq \\ &\leq \tfrac{1}{2} \cdot \phi_n(3x) + \tfrac{1}{2} \cdot \phi_n(3y) = \phi_{n+1}(x) + \phi_{n+1}(y). \end{aligned}$$

Case 4: $0 \leq y \leq \frac{1}{3} \leq x$. As $x + y \geq \frac{1}{3}$, we have

$$\begin{aligned} \phi_{n+1}(x+y) &= \tfrac{1}{2} + \tfrac{1}{2} \cdot \phi_n(3x+3y-2) \leq \\ &\leq \tfrac{1}{2} + \tfrac{1}{2} \cdot \phi_n(3x-2) + \tfrac{1}{2} \cdot \phi_n(3y) = \phi_{n+1}(x) + \phi_{n+1}(y). \end{aligned}$$

These four cases exhaust all the posibilities, so the proof is complete.

Corollary 1. *The standard Cantor function is metric preserving.*

Remark. It is not difficult to verify that $x \mapsto \frac{\phi(x)}{x}$ is not nonincreasing on $(0, +\infty)$.

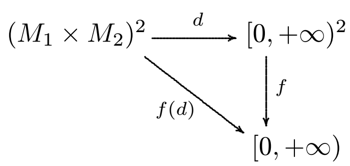
9. Metric preserving functions of several variables

There is a natural way of introducing an algebraic structure on a product of algebraic structures of the same type. For example, if $(A,\oplus)$ and $(B,\otimes)$ are groups, then $(A\times B,\odot)$, where $(a_1,b_1)\odot(a_2,b_2)=(a_1\oplus a_2,b_1\otimes b_2)$ is a group as well. The application of this method to metric spaces yields a mapping which is not a metric.

Let (M_1,d_1), (M_2,d_2) be metric spaces. It is well known that d_1+d_2, $\sqrt{d_1^2+d_2^2}$, $\max(d_1,d_2)$ are metrics on $M_1\times M_2$. In these cases we obtain new metrics as composite functions of the real functions $(x,y)\mapsto x+y$, $(x,y)\mapsto\sqrt{x^2+y^2}$, and $(x,y)\mapsto\max(x,y)$, respectively, with the "vector metric"

$$d:(M_1\times M_2)^2\to[0,+\infty)^2,\ \text{where } d((p,q),(r,s))=(d_1(p,r),d_2(q,s)).$$

We can describe these cases by the following diagram



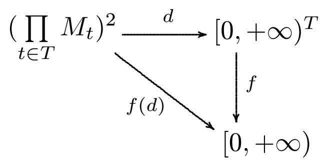
where f is a suitable function of two variables.

We shall generalize this idea.

Let T be a nonempty set of indices. Consider the indexed family $\{(M_t,d_t)\}_{t\in T}$ of metric spaces. Define $d:(\prod_{t\in T}M_t)^2\to[0,+\infty)^T$ as follows

$$d((x_t)_{t\in T},(y_t)_{t\in T})=(d_t(x_t,y_t))_{t\in T}.$$

We say that $f:[0,+\infty)^T\to[0,+\infty)$ is a *metric preserving function* if for each indexed family $\{(M_t,d_t)\}_{t\in T}$ of metric spaces the composite function $f(d)$ is a metric on the set $\prod_{t\in T}M_t$.



Let us begin by recalling that a function $f : X \to [0, +\infty)$ is said to be *subadditive* if it satisfies the inequality $f(x + y) \leq f(x) + f(y)$ whenever $x, y \in X$, where X is an additive monoid.

Let us recall that a function $f : [0, +\infty)^T \to [0, +\infty)$ is called *isotone* iff

$$f(x) \leq f(y) \text{ whenever } 0 \leq x_t \leq y_t \text{ for each } t \in T.$$

The following sufficient condition is a generalization of Theorem 1.1.

Theorem 1. (See [46].) *If $f : [0, +\infty)^T \to [0, +\infty)$ is an isotone, subadditive function vanishing exactly at the constant zero function, then it is metric preserving.*

Suppose $[0, +\infty)^T$ is ordered coordinate-wise, i.e.

$$x \leq_T y \quad \text{iff} \quad x(t) \leq y(t) \text{ for each } t \in T;$$
$$x <_T y \quad \text{iff} \quad x(y) < y(t) \text{ for each } t \in T.$$

Define a function $\Theta_T : T \to [0, +\infty)$ by $\Theta_T(t) = 0$ for each $t \in T$.

Proposition 1. *Let $f : [0, +\infty)^T \to [0, +\infty)$ be such that*

(i) $f(\Theta_T) = 0$,

(ii) $\exists a > 0\ \forall x \in [0, +\infty)^T, x \neq \Theta_T :\ a \leq f(x) \leq 2a$.

Then f is metric preserving.

The following Theorem gives a characterization of metric preserving functions. (See [5].)

Theorem 2. *Let $f : [0, +\infty)^T \to [0, +\infty)$. Then f is metric preserving iff it is a function vanishing exactly at the constant zero function and it has the following property*

$$\text{if } (a_t, b_t, c_t) \text{ is a triangle triplet for each } t \in T,$$
$$\text{then } (f((a_t)_{t\in T}), f((b_t)_{t\in T}), f((c_t)_{t\in T})) \text{ is a triangle triplet.}$$

Corollary 1. *Let $f : [0, +\infty)^T \to [0, +\infty)$ be metric preserving. Then for all $x, y \in [0, +\infty)^T$ we have*

$$(\,\forall t \in T : x(t) \leq 2y(t)\,) \Rightarrow f(x) \leq 2f(y).$$

Theorem 3. *Let $f : [0, +\infty)^T \to [0, +\infty)$ be metric preserving. Then f is continuous iff f is continuous at the point Θ_T.*

Proof. Let $\varepsilon > 0$. Then there is an open neighborhood U of the point Θ_T (in the product topology) such that for each $x \in U$ we have $f(x) < \varepsilon$. Let $V \subset U$ be a base element such that $\Theta_T \in V$, i.e. there is a nonempty finite subset F of T such that

$$V = \bigcap_{t \in F} \pi_t^{-1}([0, \gamma_t)),$$

where $\gamma_t > 0$, and π_t is the projection from $[0, +\infty)^T$ into $[0, +\infty)$, i.e. $\pi_t(x) = x(t)$ for each $x \in [0, +\infty)^T$.

Put $\gamma = \min_{t \in F} \gamma_t$. Since $\bigcap_{t \in F} \pi_t^{-1}([0, \gamma)) \subset V$, for each $x \in [0, +\infty)^T$ we have

$$(\forall t \in F : x(t) < \gamma) \Rightarrow f(x) < \varepsilon.$$

Let $x \in [0, +\infty)^T$, $x \neq \Theta_T$. Put $\delta = \gamma/2$.
Let $y \in [0, +\infty)^T$ be such that for each $t \in F$

$$|x(t) - y(t)| < \delta.$$

Define $z : T \to [0, +\infty)$ by

$$z(t) = \begin{cases} \min(\delta, x(t) + y(t)), & \text{for } t \in F, \\ x(t) + y(t), & \text{for } t \in T - F. \end{cases}$$

Evidently $(x(t), y(t), z(t))$ is a triangle triplet for each $t \in T$. Since $z(t) < \gamma$ for each $t \in F$, we obtain

$$|f(x) - f(y)| \leq f(z) < \varepsilon.$$

This shows that f is continuous at the point x.

Lemma 1. *Let $f : [0,+\infty)^T \to [0,+\infty)$ be metric preserving. If f is continuous, then*

$$\forall \varepsilon > 0 \ \exists x \in [0,+\infty)^T, \Theta_T <_T x : f(x) < \varepsilon.$$

Proof. Let $\varepsilon > 0$. Since f is continuous at the point Θ_T, there is a neighbourhood U of Θ_T (in the product topology) such that for all $x \in U$ we have $f(x) < \varepsilon$. Then there are $\delta > 0$ and a nonempty finite subset F of T such that

$$\bigcap_{t \in F} \pi_t^{-1}([0,\delta)) \subset U.$$

Define a function $x : T \to [0,+\infty)$ by $x(t) = \delta/2$ for each $t \in T$.
Since $x \in U$, we have $f(x) < \varepsilon$.

Proposition 2. *Let T be a finite set. Let $f : [0,+\infty)^T \to [0,+\infty)$ be metric preserving. Then f is continuous iff*

$$\forall \varepsilon > 0 \ \exists x \in [0,+\infty)^T, \Theta_T <_T x : \ f(x) < \varepsilon.$$

Proof. Let $\varepsilon > 0$. Then there is $a \in [0,+\infty)^T$ such that $\Theta_T <_T a$ and $f(a) < \varepsilon/2$. Put

$$U = \bigcap_{t \in T} \pi_t^{-1}([0, \min_{t \in T} a(t))).$$

By Corollary 1 for all $x \in U$ we obtain $f(x) \leq 2f(a) < \varepsilon$, therefore f is continuous at the point Θ_T.

The following example shows that the assumption "$\Theta_T <_T x$" in Proposition 2 cannot be replaced by the assumption "$x \neq \Theta_T$".

Example 1. Let $f : [0,+\infty)^2 \to [0,+\infty)$ be defined as follows:

$$f(x,y) = \begin{cases} \min(1,y) & \text{for } \ x = 0, \\ 1 & \text{for } x \neq 0. \end{cases}$$

Then f is metric preserving and discontinuous, but

$$\forall \varepsilon > 0 \ \exists x \in [0,+\infty)^2, x \neq \Theta_2 : f(x) < \varepsilon$$

(for example $x = (0, \min(\frac{1}{2}, \frac{\varepsilon}{2}))$).

Corollary 2. *Let T be a finite set. Let f be metric preserving. Then f is discontinuous if and only if*

$$\exists \eta > 0 \ \forall x \in [0,+\infty)^T, \Theta_T <_T x : f(x) \geq \eta.$$

10. Metrization of the product topology

Consider an indexed family $\{(M_t, d_t)\}_{t\in T}$ of metric spaces. Denote by $\mathcal{T}_\sqcap$ the product topology on $\prod_{t\in T} M_t$.

For each metric preserving function f denote by $\mathcal{T}_f$ the topology on $\prod_{t\in T} M_t$ generated by the metric $f(d)$. A natural question arises whether we can investigate metrizability of the product topology $\mathcal{T}_\sqcap$ by the metric $f(d) = f \circ d$. The results in this section are extracted from [5].

Lemma 1. *Let $f : [0, +\infty)^T \to [0, +\infty)$ be metric preserving. Then*

$$\mathcal{T}_\sqcap \subset \mathcal{T}_f.$$

Proof. Let $t \in T$. Let $B_{d_t}(x_t, \varepsilon)$ be an open ball in the metric space M_t. Let $x \in \pi_t^{-1}(B_{d_t}(x_t, \varepsilon))$ be such that $x(t) = x_t$ (where π_t is the projection from $\prod_{t\in T} M_t$ into M_t). Define $a : T \to [0, +\infty)$ by $a(t) = 2\varepsilon$ and $a(i) = 0$ for each $i \in T - \{t\}$. Put $\delta = f(a)/2$. Let $y \in B_{f(d)}(x, \delta)$. Then

$$f(d(x, y)) < \delta = f(a)/2.$$

By Corollary 9.1 we have

$$(\,\forall i \in T : a(i) \le 2d_i(x(i), y(i))\,) \Rightarrow f(a) \le 2f(d(x, y)),$$

or equivalently

$$f(d(x, y)) < f(a)/2 \Rightarrow (\exists j \in T : d_j(x(j), y(j)) < a(j)/2).$$

By definition of a we obtain $j = t$, which yields $d_t(x(t), y(t)) < a(t)/2 = \varepsilon$, i.e. $y \in \pi_t^{-1}(B_{d_t}(x_t, \varepsilon))$.

This shows that $\mathcal{T}_\sqcap \subset \mathcal{T}_f$.

Let $S \subset T$ be a nonempty set. Define $\iota_{S,T} : [0, +\infty)^S \to [0, +\infty)^T$ by

$$(\iota_{S,T}(a))(x) = \begin{cases} a(t), & \text{for } t \in S, \\ 0, & \text{for } t \in T - S, \end{cases} \qquad \text{for each } a \in [0, +\infty)^S.$$

Put $H = \{t \in T : \text{ the metric space } (M_t, d_t) \text{ is not discrete}\}$.

Proposition 1. *Let F be a nonempty subset of T such that $H \subset F$ and $T-F$ is finite. Let $f : [0,+\infty)^T \to [0,+\infty)$ be metric preserving. If $f(\iota_{F,T})$ is continuous, then $\mathcal{T}_\sqcap = \mathcal{T}_f$.*

Proof. We show that $\mathcal{T}_f \subset \mathcal{T}_\sqcap$. Let $x \in \prod_{t\in T} M_t$ and $\varepsilon > 0$. Since $f(\iota_{F,T})$ is continuous at the point Θ_F, there is a nonempty finite subset K of F and $\gamma > 0$ such that for each $y \in [0,+\infty)^F$

$$(\forall t \in K : y(t) < \gamma) \Rightarrow f(\iota_{F,T}(y)) < \varepsilon.$$

Since $T - F$ is finite, there is $\beta > 0$ such that for each $t \in T - F$

$$B_{d_t}(x(t), \beta) = \{x(t)\}.$$

Put $\delta = \min(\beta, \gamma)$, $L = K \cup (T - F)$, and $V = \bigcap_{t\in L} \pi_t^{-1}(B_{d_t}(x(t), \delta))$. Then V is a neighborhood of x in the $\mathcal{T}_\sqcap$.
It is not difficult to verify that $V \subset B_{f(d)}(x, \varepsilon)$.

Corollary 1. *Let $f : [0,+\infty)^T \to [0,+\infty)$ be a continuous metric preserving function. Then $\mathcal{T}_\sqcap = \mathcal{T}_f$.*

Put $I = \{t \in T :$ the metric space (M_t, d_t) is bounded$\}$.

Proposition 2. *Suppose that $H \cap I$ is finite. Let $f : [0,+\infty)^T \to [0,+\infty)$ be metric preserving. If $\mathcal{T}_\sqcap = \mathcal{T}_f$, then $f(\iota_{H,T})$ is continuous.*

Proof. Since $f(\iota_{H,T}) : [0,+\infty)^H \to [0,+\infty)$ is metric preserving, it is sufficient to prove that it is continuous at the point Θ_H. Let $x \in \prod_{t\in T} M_t$ be such that, for all $t \in H$, $x(t)$ is an accumulation point of the set M_t. Let $\varepsilon > 0$. Then $B_{f(d)}(x, \frac{\varepsilon}{2}) \in \mathcal{T}_f \subset \mathcal{T}_\sqcap$, hence

$$\exists K \subset T, K \neq \emptyset \text{ finite } \exists \gamma > 0 : \bigcap_{t\in K} \pi_t^{-1}(B_{d_t}(x(t), \gamma)) \subset B_{f(d)}(x, \tfrac{\varepsilon}{2}).$$

Let F be a nonempty finite set such that $H \cap (K \cup I) \subset F \subset H$. Let $t \in F$. Since $x(t)$ is an accumulation point of M_t, there exists $y_t \in M_t$ with $0 < d_t(x(t), y_t) < \gamma$. Put $\delta = \min_{t\in F} d_t(x(t), y_t)$. Let $z \in [0,+\infty)^H$ be such that $z \in \bigcap_{t\in F} \pi_t^{-1}([0, \delta))$. Then

$$\forall t \in H - F \ \exists y_t \in M_t : z(t) \leq d_t(x(t), y_t).$$

Define a mapping $y : T \to \bigcup_{t\in T} M_t$ by $y(t) = y_t$ for $t \in H$, $y(t) = x(t)$ for $t \in T - H$. Then $y \in \bigcap_{t\in K} \pi_t^{-1}(B_{d_t}(x(t), \gamma))$ and by Corollary 9.1

$$(f(\iota_{H,T}(z)) \leq 2f(d(x,y)) < \varepsilon.$$

This shows that $f(\iota_{H,T})$ is continuous at the point Θ_H.

Corollary 1. *Let $f : [0, +\infty)^T \to [0, +\infty)$ be metric preserving. In the case of $(M_t, d_t) = (\mathbb{R}, e)$ for each $t \in T$ (where e is the Euclidean metric), $\mathcal{T}_\sqcap = \mathcal{T}_f$ iff f is continuous.*

Corollary 2. *Let T be finite. Let $f : [0, +\infty)^T \to [0, +\infty)$ be metric preserving. Then $\mathcal{T}_\sqcap = \mathcal{T}_f$ iff $f(\iota_{H,T})$ is continuous.*

The following example shows that the assumption "T is finite" in Corollary 2 cannot be omitted.

Example 1. Consider (for each $n \in \mathbb{N}$) $M_n = [0, 1/n]$ with the usual metric $e_n(u, v) = |u - v|$.

Define $f : [0, +\infty)^{\mathbb{N}} \to [0, +\infty)$ by $f(x) = \sup_{n \in \mathbb{N}}(\min(1, x(n)))$ for all $x \in [0, +\infty)^{\mathbb{N}}$. Then we can verify that f is metric preserving, $\mathcal{T}_\sqcap = \mathcal{T}_f$ but f is not continuous.

Theorem 1. (See [5].) *Let $f : [0, +\infty)^T \to [0, +\infty)$ be metric preserving. Then $\mathcal{T}_\sqcap = \mathcal{T}_f$ iff*

$$\forall\ \varepsilon > 0\ \exists\ F \subset T \text{ finite } \exists\ \delta > 0\ \forall\ \alpha \in \mathbb{N}^T\ \exists\ a \in [0, +\infty)^T :$$

(i) $$\forall\ t \in T - (I \cup F) : a(t) \geq \alpha(t),$$

(ii) $$\forall\ t \in I - F : a(t) \geq \operatorname{diam} M_t,$$

(iii) $$\forall\ t \in F \cap H : a(t) \geq \delta,$$

(iv) $$f(a) < \varepsilon.$$

Proof. Necessity. Choose $x \in \prod_{t \in T} M_t$ such that $x(t)$ is an accumulation point of the set M_t for each $t \in H$. Let $\varepsilon > 0$. Since $\mathcal{T}_\sqcap = \mathcal{T}_f$,

$$\exists F \subset T, F \neq \emptyset \text{ finite } \exists \gamma > 0 : \bigcap_{t \in F} \pi_t^{-1}(B_{d_t}(x(t), \gamma)) \subset B_{f(d)}(x, \varepsilon/4).$$

Let $t \in F \cap H$. Then there is $y_t \in M_t$ such that $0 < d_t(x(t), y_t) < \gamma$. Put

$$\delta = \begin{cases} \min_{t \in F \cap H} d_t(x(t), y_t), & \text{if } F \cap H \neq \emptyset \\ 1, & \text{if } F \cap H = \emptyset. \end{cases}$$

Let $\alpha \in \mathbb{N}^T$. Let $t \in T - (I \cup F)$. Then there is $y_t \in M_t$ such that $d_t(x(t), y_t) \geq \alpha(t)$. Let $t \in I - F$. If $\operatorname{diam} M_t > 0$, there exists $y_t \in M_t$ such that

$$d_t(x(t), y_t) > \tfrac{1}{4} \cdot \operatorname{diam} M_t.$$

If $\operatorname{diam} M_t = 0$, put $y_t = x(t)$. For each $t \in F - H$ put $y_t = x(t)$. Define a mapping $y : T \to \bigcup_{t\in T} M_t$ by $y(t) = y_t$ for all $t \in T$. Put $a = 4d(x,y)$. Then

$$f(a) \le 4 \cdot f(d(x,y)) < 4 \cdot \varepsilon/4 = \varepsilon.$$

Sufficiency. Let $x \in \prod_{t\in T} M_t$ and $\varepsilon > 0$. Then there exists a finite set $F \subset T$ such that

$$\exists \delta > 0\ \forall \alpha \in \mathbb{N}^T\ \exists a \in [0,+\infty)^T :$$

(i) $$\forall t \in T - (I \cup F) :\ a(t) \ge \alpha(t),$$

(ii) $$\forall t \in I - F : a(t) \ge \operatorname{diam} M_t,$$

(iii) $$\forall t \in F \cap H : a(t) \ge \delta,$$

(v) $$f(a) < \varepsilon/2.$$

Since $F - H$ is finite there is $\gamma > 0$ such that

$$\forall t \in F - H\ \forall y \in M_t, y \ne x(t) : d_t(x(t), y) \ge \gamma.$$

Let K be a nonempty finite set such that $F \subset K \subset T$. Put

$$V = \bigcap_{t\in K} \pi_t^{-1}(B_{d_t}(x(t), \min(\gamma,\delta))).$$

Let $y \in V$. Let $t \in (T - ((I \cup F))$. Then there exists a positive integer n_t such that $d_t(x(t), y(t)) \le n_t$. Define a mapping $\alpha : T \to \mathbb{N}$ by

$$\alpha(t) = \begin{cases} n_t, & \text{for each } t \in T - (I \cup F) \\ 1, & \text{otherwise.} \end{cases}$$

Then there is $a \in [0,+\infty)^T$ such that (i), (ii), (iii) and (v) hold. It is not difficult to verify that $d(x,y) \le_T a$. By Corollary 9.1 we obtain

$$f(d(x,y)) \le 2f(a) < 2 \cdot \varepsilon/2 = \varepsilon, \text{ i.e. } y \in B_{f(d)}(x,\varepsilon).$$

This shows that $\mathcal{T}_f \subset \mathcal{T}_\sqcap$.

This Theorem shows that this metrizability depends on boundedness and discretness of factors only.

11. Metrization of the product uniformity

Consider an indexed family $\{(M_t, d_t)\}_{t\in T}$ of metric spaces. Denote by $\mathcal{U}_\sqcap$ the product uniformity on $\prod_{t\in T} M_t$.

For each metric preserving function f denote by $\mathcal{U}_f$ the uniformity on $\prod_{t\in T} M_t$ generated by the metric $f(d)$. A natural question arises whether we can investigate metrizability of the product uniformity $\mathcal{U}_\sqcap$ by the metric $f(d)$. The results in this section are extracted from [6].

Lemma 1. *Let $f : [0, +\infty)^T \to [0, +\infty)$ be metric preserving. Then*

$$\mathcal{U}_\sqcap \subset \mathcal{U}_f.$$

Proof. Let $U \in \mathcal{U}_\sqcap$. Then there is a finite nonempty subset F of T and $\varepsilon > 0$ such that

$$\bigcap_{t\in F} (\pi_t \times \pi_t)^{-1}(d_t^{-1}([0, \varepsilon))) \subset U.$$

Define a mapping $\eta : F \to [0, +\infty)^T$ as follows

$$(\eta(t))(i) = \begin{cases} 2\varepsilon, & \text{if } i = t, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{for each } t \in F.$$

Let $t \in F$. Put $\delta_t = f(\eta(t))/2$ and $W_t = (f(d))^{-1}([0, \delta_t))$. We show that $W_t \subset d_t^{-1}([0, \varepsilon))$. Let $(x, y) \in W_t$. Then $f(d(x, y)) < \delta_t = f(\eta(t))/2$, therefore by Corollary 9.1 we obtain

$$d_t(x(t), y(t)) < (\eta(t))(t)/2 = \varepsilon.$$

This shows that $\bigcap_{t\in F} W_t \subset U$. Evidently $\bigcap_{t\in F} W_t \in \mathcal{U}_f$.

Proposition 1. *Let $f : [0, +\infty)^T \to [0, +\infty)$ be metric preserving. Suppose that f is continuous. Then $\mathcal{U}_\sqcap = \mathcal{U}_f$.*

Proof. Let $U \in \mathcal{U}_f$. Then there is $\varepsilon > 0$ such that $(f(d))^{-1}([0, \varepsilon)) \subset U$. Since f is continuous at the point Θ_T, there exists a finite nonempty subset F of T and exists $\gamma > 0$ such that

$$\forall a \in [0, +\infty)^T : (\forall t \in F : a(t) < \gamma) \Rightarrow f(a) < \varepsilon.$$

Put $V = \bigcap_{t\in F} (\pi_t \times \pi_t)^{-1}(d_t^{-1}([0, \gamma)))$. Evidently $V \in \mathcal{U}_\sqcap$. We show that

$$V \subset (f(d))^{-1}([0, \varepsilon)).$$

Let $(x, y) \in V$. Then $d_t(x(t), y(t)) < \gamma$ for all $t \in F$, which yields

$$f(d(x, y)) < \varepsilon.$$

This shows that $V \subset U$.

Put $S = \{t \in T : \text{ the metric space } (M_t, d_t) \text{ is not uniformly discrete}\}$.

Theorem 1. (See [6].) *Let $f : [0,+\infty)^T \to [0,+\infty)$ be metric preserving. Then $\mathcal{U}_\sqcap = \mathcal{U}_f$ iff*

$$\forall\ \varepsilon > 0\ \exists\ F \subset T \text{ finite } \exists\ \delta > 0\ \forall\ \alpha \in \mathbb{N}^T\ \exists\ a \in [0,+\infty)^T :$$

(i) $$\forall\ t \in T - (I \cup F) : a(t) \geq \alpha(t),$$

(ii) $$\forall\ t \in I - F : a(t) \geq \operatorname{diam} M_t,$$

(iii) $$\forall\ t \in F \cap S : a(t) \geq \delta,$$

(iv) $$f(a) < \varepsilon.$$

Proof. Necessity. Let $\varepsilon > 0$. Since $\mathcal{U}_f \subset \mathcal{U}_\sqcap$, we have

$$(f(d))^{-1}([0,\varepsilon/2)) \in \mathcal{U}_\sqcap.$$

Thus there is a finite nonempty subset F of T and $\gamma > 0$ such that

$$\bigcap_{t\in F} (\pi_t \times \pi_t)^{-1}(d_t^{-1}([0,\gamma))) \subset (f(d))^{-1}([0,\varepsilon/2)).$$

Let $t \in F \cap S$. Then there are $u_t, v_t \in M_t$ such that

$$0 < d_t(u_t, v_t) < \gamma.$$

Put $\delta = \min\{d_t(u_t,v_t) : t \in F \cap S\}$ (in the case of $F \cap S = \emptyset$ let $\delta > 0$ be arbitrary). Let $\alpha \in \mathbb{N}^T$. Let $t \in T - I$. Then there are $p_t, q_t \in M_t$ such that

$$d_t(p_t, q_t) \geq \alpha(t).$$

Put $J = \{t \in I : \operatorname{diam} M_t > 0\}$. Let $t \in J$. Then there are $r_t, s_t \in M_t$ such that

$$d_t(r_t, s_t) > \tfrac{1}{2} \cdot \operatorname{diam} M_t.$$

Let $t \in T$. Since M_t is a nonempty set, choose an arbitrary element $w_t \in M_t$.

Define the mappings $x, y : T \to \bigcup_{t\in T} M_t$ as follows

$$x(t) = \begin{cases} u_t, \\ p_t, \\ r_t, \\ w_t, \end{cases} \qquad y(t) = \begin{cases} v_t, & \text{for } t \in F \cap S, \\ q_t, & \text{for } t \in T - (I \cup F), \\ s_t, & \text{for } t \in J - F, \\ w_t, & \text{for } t \in [I - (J \cup F)] \cup (F - S). \end{cases}$$

Put $a = 2d(x, y)$. Now we show that a satisfies the conditions (i)–(iv).

(i): Let $t \in T-(I\cup F)$. Then $a(t) = 2\cdot d_t(x(t), y(t)) = 2\cdot d_t(p_t, q_t) \geq \alpha(t)$.

(ii): Let $t \in I - (J \cup F)$. Then $a(t) = 2 \cdot d_t(w_t, w_t) = 0 = \operatorname{diam} M_t$.
Let $t \in J-F$. Then $a(t) = 2\cdot d_t(r_t, s_t) > 2\cdot \frac{1}{2}\cdot \operatorname{diam} M_t = \operatorname{diam} M-t$.
This shows that $a(t) \geq \operatorname{diam} M_t$ for all $t \in I - F$.

(iii): Let $t \in F \cap S$. Then $a(t) = 2 \cdot d_t(u_t, v_t) \geq \delta$.

(iv): Let $t \in F \cap S$. Then $d_t(x(t), y(t)) = d_t(u_t, v_t) < \gamma$.
Let $t \in F - S$. Then $d_t(x(t), y(t)) = d_t(w_t, w_t) = 0 < \gamma$.
Therefore $d_t(x(t), y(t)) < \gamma$ for each $t \in F$, i.e.

$$(x, y) \in \bigcap_{t\in F} (\pi_t \times \pi_t)^{-1}(d_t^{-1}([0, \gamma))).$$

Then by Corollary 9.1 we obtain

$$f(a) \leq 2 \cdot f(d(x, y)) < 2 \cdot \frac{\varepsilon}{2} = \varepsilon.$$

Sufficiency. By Lemma 1 it suffices to prove that $\mathcal{U}_f \subset \mathcal{U}_\sqcap$. Let $U \in \mathcal{U}_f$. Then there is $\varepsilon > 0$ such that

$$(f(d))^{-1}([0, 2\varepsilon)) \subset U.$$

Then by the hypotheses we have

$$\exists F \subset T, F \neq \emptyset \text{ finite } \exists \delta > 0\ \forall \alpha \in \mathbb{N}^T\ \exists a \in [0, +\infty)^T : \text{(i)–(iv)}.$$

Let $\gamma > 0$ be such that

$$d_t^{-1}((0, \gamma)) = \emptyset \text{ for each } t \in F - S.$$

Put

$$A = \bigcap_{t\in F} (\pi_t \times \pi_t)^{-1}(d_t^{-1}([0, \min(\gamma, \delta)))).$$

Evidently $A \in \mathcal{U}_\sqcap$. Let $(x, y) \in A$. Then

$$d_t(x(t), y(t)) < \min(\gamma, \delta) \text{ for each } t \in F.$$

Let $t \in T - (I \cup F)$. Then there is a positive integer n_t such that

$$d_t(x(t), y(t)) \leq n_t.$$

Define a mapping $\alpha : T \to \mathbb{N}$ by

$$\alpha(t) = \begin{cases} n_t, & \text{for } t \in T - (I \cup F), \\ 1, & \text{otherwise.} \end{cases}$$

Then there is $a \in [0, +\infty)^T$ satisfying (i)–(iv). We show that $d(x, y) \leq a$:

$$d_t(x(t), y(t)) \leq \begin{cases} \operatorname{diam} M_t & \text{for each } t \in I - F, \\ \delta & \text{for each } t \in F \cap S, \\ \alpha(t) & \text{for each } t \in T - (I \cup F), \\ 0 & \text{for each } t \in F - S, \end{cases}$$

which yields $d_t(x(t), y(t)) \leq a(t)$ for each $t \in T$. Therefore by Corollary 9.1 we obtain

$$f(d(x, y)) \leq 2f(a) < 2\varepsilon,$$

i.e. $(x, y) \in (f(d))^{-1}([0, 2\varepsilon))$.

This show that $A \subset (f(d))^{-1}([0, 2\varepsilon)) \subset U$, therefore $U \in \mathcal{U}_{\sqcap}$.

12. Topologies on a product of metric spaces

The results in this section are extracted from [18].

Proposition 1. *Let $Q \neq \emptyset$ be finite. Let $f : [0, +\infty)^Q \to [0, +\infty)$ be metric preserving. Then f is continuous if and only if $f(\iota_{\{q\},Q})$ is continuous for each $q \in Q$.*

Proof. One part of the proof follows from the fact that $\iota_{S,T}$ is continuous for each $S \subset T$ (see [8], p. 59). For the second part, let $\varepsilon > 0$, $q \in Q$. Since $f(\iota_{\{q\},Q})$ is continuous,

$$\exists x_q > 0 : (f\iota_{\{q\},Q}(x_q)) < \varepsilon/(\operatorname{card} Q).$$

Put

$$a = \sum_{q \in Q} \iota_{\{q\},Q}(x_q).$$

Thus $a \in [0, +\infty)^Q$, $\Theta_Q <_Q a$, and

$$f(a) \leq \sum_{q \in Q} f(\iota_{\{q\},Q}(x_q)) < \varepsilon.$$

Then by Proposition 9.2 the function f is continuous.

For each metric preserving function $f : [0, +\infty)^T \to [0, +\infty)$ put

$$F(f) = \{t \in T : f(\iota_{\{t\},T}) \text{ is continuous}\}.$$

Corollary 1. *Let T be a nonempty set. Let $f : [0, +\infty)^T \to [0, +\infty)$ be metric preserving. Then $f(\iota_{S,T})$ is continuous iff $S \subset F(f)$.*

The following example shows that the condition "finite" in Proposition 1 cannot be omitted.

Example 1. Let $P \neq \emptyset$. Define a mapping $f : [0, +\infty)^P \to [0, +\infty)$ as follows

$$f(x) = \sup\{\min(1, x_t) : t \in P\}.$$

Then f is metric preserving, $F(f) = P$, and f is continuous iff P is finite.

Define a function $j_T : [0, +\infty)^T \to [0, +\infty)$ by

$$j_T(x) = \begin{cases} 0 & \text{for } x = \Theta_T, \\ 1 & \text{for } x \neq \Theta_T. \end{cases}$$

Proposition 2. *Let the set T be finite. Let $h : [0,+\infty)^T \to [0,+\infty)$ be a continuous metric preserving functions. Let S be a nonempty finite subset of T. Define a mapping $h_S : [0,+\infty)^T \to [0,+\infty)$ as follows*

$$h_S(x) = \begin{cases} h(x)/(1+h(x)) & \textit{for } x \in \operatorname{Im}(i_{S,T}), \\ 1 & \textit{otherwise.} \end{cases}$$

Then h_S is metric preserving and $F(h_S) = S$.

Proof. Let $x \in [0,+\infty)^T$. Then $h_S(x) = 0 \Leftrightarrow h(x) = 0 \Leftrightarrow x = \Theta_T$.
Let $x, y, z \in [0,+\infty)^T$, $x \leq_T y+z$, $y \leq_T x+z$, $z \leq_T x+y$. Since h is metric prreserving, we have

$$h(x) \leq h(y) + h(z).$$

If $h_S(y)+h_S(z) < 1$, then $x, y, z \in \operatorname{Im}(\iota_{S,T})$, thus $h_S(x) = h(x)/(1+h(x)) \leq h(y)/(1+h(y)) + h(z)/(1+h(z)) = h_S(y) + h_S(z)$. If $h_S(y) + h_S(z) \geq 1$, then $h_S(x) \leq 1 \leq h_S(y) + h_S(z)$. This shows that h_S is metric preserving.
Since h is continuous, $h_S(\iota_{S,T}) = \dfrac{h(\iota_{S,T})}{1+h(\iota_{S,T})}$ is continuous. Thus

$$S \subset F(h_S).$$

Since for each $t \in T - S$ the function $h_S(\iota_{\{t\},T}) = j_T(\iota_{\{t\},T})$ is not continuous, we have $T - S \subset T - F(h_S)$.

Consider an indexed family $\{(M_t, d_t)\}_{t \in T}$ of metric spaces (where $T \neq \emptyset$ is a set of indices). Put

$$\mathcal{L} = \{\mathcal{T}_f; f : [0,+\infty)^T \to [0,+\infty) \text{ is a metric preserving function}\}.$$

Proposition 3. *Let $f, g : [0,+\infty)^T \to [0,+\infty)$ be metric preserving. Suppose that $\mathcal{T}_f \subset \mathcal{T}_g$. Then*

$$F(f) \supset F(g) \cap H.$$

Proof. Let $t \in F(g) \cap H$. Let $\varepsilon > 0$. Select $a \in \prod_{t \in T} M_t$ such that $a(t)$ is an accumulation point of M_t. Since $\mathcal{T}_f \subset \mathcal{T}_g$, there exists $\delta > 0$ such that

$$B_{g(d)}(a, 2\delta) \subset S_{f(d)}(a, \varepsilon).$$

Since $g(\iota_{\{t\},T})$ is continuous, we have

$$\exists y > 0 : g(\iota_{\{t\},T}(y)) < \delta.$$

Let $q \in M_t$ such that $0 < d_t(a(t), q) < y$. Define a mapping $b : T \to \bigcup_{t\in T} M_t$ as follows

$$b(s) = \begin{cases} q & \text{for } s = t, \\ a(s) & \text{otherwise.} \end{cases}$$

Put $x = d_t(a(t), b(t))$. Since $(g\iota_{\{t\},T})$ is metric preserving and $x \le y$, we have $g(d(a,b)) = g(\iota_{\{t\},T}(x)) \le 2 \cdot g(\iota_{\{t\},T})(y)) < 2\delta$. Thus $b \in S_{g(d)}(a, 2\delta)$. Then we have $f(\iota_{\{t\},T})(x)) = f(d(a,b)) < \varepsilon$. This shows that the function $f(\iota_{\{t\},T})$ is continuous.

In the following it will be proved that if T is finite, then the topologies generated by metric preserving functions are determined by subsets of the set of all indices t such that the metric spaces (M_t, d_t) are not discrete.

Theorem 1. *Let the set T be finite. Let $f, g : [0,+\infty)^T \to [0,+\infty)$ be metric preserving. Then $\mathcal{T}_f \subset \mathcal{T}_g$ iff $F(f) \supset F(g) \cap H$.*

Proof. One part of the proof follows from Proposition 3.
For the second part, let $a \in \prod_{t\in T} M_t$, $\varepsilon > 0$. We show that

$$\exists \delta > 0 : B_{g(d)}(a, \delta) \subset B_{f(d)}(a, \varepsilon).$$

Let $\gamma > 0$ such that

$$\forall t \in T - H \;\forall b \in [0,+\infty)^T : (d_t(a(t), b(t)) < \gamma) \Rightarrow a(t) = b(t).$$

Let $\eta > 0$ such that

$$\forall t \in T - F(g) \;\forall x > 0 : g(\iota_{\{t\},T}(x)) \ge \eta.$$

Let $t \in F(f)$. Since $f(\iota_{\{t\},T})$ is continuous, there exists $x_t > 0$ such that

$$f(\iota_{\{t\},T}(x_t)) < \varepsilon/(2 \operatorname{card} T).$$

Put

$$\delta_t = g(\iota_{\{t\},T}(x_t))/2.$$

For each $t \in T - F(f)$ put $x_t = 0$. For each $t \in T$ put

$$\gamma_t = g(\iota_{\{t\},T}(\gamma))/2.$$

Put

$$\delta = \min(\{\delta_t : t \in F(f)\} \cup \{\gamma_t : t \in T\} \cup \{\eta/2\}).$$

Let $b \in B_{g(d)}(a, \delta)$, $t \in F(f)$. Since $2g(d(a,b) < 2\delta \leq g(\iota_{\{t\},T}(x_t))$, we have

$$2d_t(a(t), b(t)) < x_t.$$

Let $t \in T - H$. Since $2g(d(a,b)) \leq 2\delta \leq g(\iota_{\{t\},T}(y_t))$, we have

$$2d_t(a(t), b(t)) < \gamma.$$

Therefore $a(t) = b(t)$.

Let $t \in T - F(g)$. Put

$$u = d_t(a(t), b(t)).$$

Since $\iota_{\{t\},T}(u) \leq_T 2 \cdot d(a,b))$, we have

$$g(\iota_{\{t\},T}(u)) \leq 2 \cdot g(d(a,b)) < 2\delta \leq \eta.$$

Thus $a(t) = b(t)$. Therefore we obtain

$$f(d(a,b)) \leq 2 \cdot f\left(\sum_{t\in T} \iota_{\{t\},T}(x_t)\right) \leq 2 \cdot \sum_{t\in T} f(\iota_{\{t\},T}(x_t)) < \varepsilon.$$

This shows that $b \in B_{f(d)}(a, \varepsilon)$.

Corollary 1. *Let the set T be finite. Let $f, g : [0, +\infty)^T \to [0, +\infty)$ be metric preserving. Then $\mathcal{T}_f = \mathcal{T}_g$ iff $H \cap F(f) = H \cap F(g)$.*

The following example shows that the condition “finite” in Proposition 3 cannot be omitted.

Example 2. Let P be an infinite set. Let $a : P \to \mathbb{N}$ be a surjection. Define a mapping $g : [0, +\infty)^P \to [0, +\infty)$ as follows

$$g(x) = \sup\{\min(1, a(t) \cdot x(t)) : t \in P\}.$$

It is not difficult to verify that g is metric preserving. Consider the metric preserving function f from Example 1. Consider the indexed family of metric spaces $\{(M_t, d_t)\}_{t \in P}$ given by $M_t = \mathbb{R}$, $d_t(x, y) = |x - y|$ for each $t \in P$. Evidently $B_{g(d)}(\Theta_P, 1) \in \mathcal{T}_g$. We prove that $B_{g(d)}(\Theta_P, 1) \notin \mathcal{T}_f$. Since for every constant function $u \in [0, +\infty)^P$, $u \neq \Theta_P$ we have $g(u) = 1$, for every $\varepsilon > 0$ we obtain

$$B_{f(d)}(\Theta_P, \varepsilon) \not\subset B_{g(d)}(\Theta_P, 1).$$

Thus $B_{g(d)}(\Theta_P, 1)$ is not the neighbourhood of the point Θ_P in $\mathcal{T}_f$. This shows that $B_{g(d)}(\Theta_P, 1) \notin \mathcal{T}_f$. Then $\mathcal{T}_g \not\subset \mathcal{T}_f$, but $F(f) = F(g) = P$.

Proposition 4. *Let the set T be finite. Let $h : [0, +\infty)^T \to [0, +\infty)$ be a continuous metric preserving function. Put $h_\emptyset = j_T$. Then*

$$\mathcal{L} = \{\mathcal{T}_{h_S} : S \subset H\}.$$

Proof. Let $f : [0, +\infty)^T \to [0, +\infty)$ be a metric preserving function. Put $S = H \cap F(f)$. Then we have $H \cap F(h_S) = H \cap S = H \cap F(f)$. Therefore $\mathcal{T}_f = \mathcal{T}_{h_S}$.

It is not difficult to prove that the partially ordered set $(\mathcal{L}, \subset)$ is a lattice.

Theorem 2. *The lattices $(\mathcal{L}, \subset)$ and $(\exp H, \subset)$ are dually isomorphic.*

Proof. Define a mapping $\Omega : \mathcal{L} \to \exp H$ by

$$\Omega(\mathcal{T}_f) = H \cap F(f).$$

Then the mapping Ω is a dual isomorphism.

A characterization of the lattice of topologies $\mathcal{T}_f$ on an infinite products of metric spaces is an open question.

13. Isotone metric preserving functions

Lassak [35] investigated metric preserving functions of the form $x \mapsto ||x||$, where $||\cdot||$ is a norm in the space $\mathbb{R}^n$.

Proposition 1. (See [35], [29].) *Let $||\cdot||$ be a norm in $\mathbb{R}^n$. Then the function $x \mapsto ||x||$ is metric preserving iff it is isotone.*

There is a norm in $\mathbb{R}^2$ which is not isotone
(for example $||(x,y)|| = \sqrt{x^2+y^2-xy}$).

A different approach can be found in Aumann's monograph [1]. For each function $f : [0,+\infty)^n \to [0,+\infty)$ define the functions
$f_i : [0,+\infty) \to [0,+\infty)$ $(i = 1,2,\ldots,n)$ as follows

$$\begin{aligned}
f_1(x) &= f(x,0,0,\ldots,0),\\
f_2(x) &= f(0,x,0,\ldots,0),\\
&\vdots\\
f_n(x) &= f(0,\ldots,0,0,x),\\
f_\triangle(x) &= f(x,x,\ldots,x,x).
\end{aligned}$$

We say that a function $f : [0,+\infty)^n \to [0,+\infty)$ is an *Aumann function* iff

(1) f is isotone,
(2) f is subadditive,
(3) $f_1(x) = f_2(x) = \cdots = f_n(x) = x$ for each $x \in [0,+\infty)$.

The following three functions are examples of such functions:

$$f_1(x) = \sqrt{\sum_{i=1}^n x_i^2}, \quad f_2(x) = \sum_{i=1}^n x_i, \quad f_3(x) = \max_{1\le i\le n}\{x_i\}.$$

The property (3) determines Aumann functions on the border of their domain. The greatest variability of these functions we can expect on the diagonal.

Proposition 2. *Let f be an Aumann function. Then the function $f_\triangle$ has the following properties*

(i) *$f_\triangle$ is nondecreasing,*
(ii) *$f_\triangle$ is subadditive, and*
(iii) *$x \le f_\triangle(x) \le nx$ for each $x \in [0,+\infty)$.*

A natural question is how much the properties (i)–(iii) determine Aumann functions on the diagonal. The answer is provided by the following theorem.

Theorem 1. (See [50].) *Let* $\varphi : [0, +\infty) \to [0, +\infty)$ *be a function satisfying properties (i)–(iii). Then there is an Aumann function* f *such that* $\varphi = f_\vartriangle$.

Proof. It is sufficient to define this function f by

$$(*) \qquad f(x) = \max\left\{ \max_{1 \le i \le n} \{a_i\}, f\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)\right\}.$$

It is not difficult to verify that the function f has the required properties.

The following example shows that there is an Aumann function which is not a norm in $\mathbb{R}^n$.

Example 1. Let f has the form (*), where

$$\varphi(x) = \begin{cases} 2x, & \text{if } 0 \le x \le 0.5, \\ 1, & \text{if } 0.5 \le x \le 1, \\ x, & \text{otherwise.} \end{cases}$$

This function is not homogeneous (e.g. $\frac{3}{4} f(1,1,\ldots,1) = \frac{3}{4} \ne 1 = f(\frac{3}{4}, \frac{3}{4}, \ldots, \frac{3}{4})$). (See Fig. 18 for the case $n = 2$.)

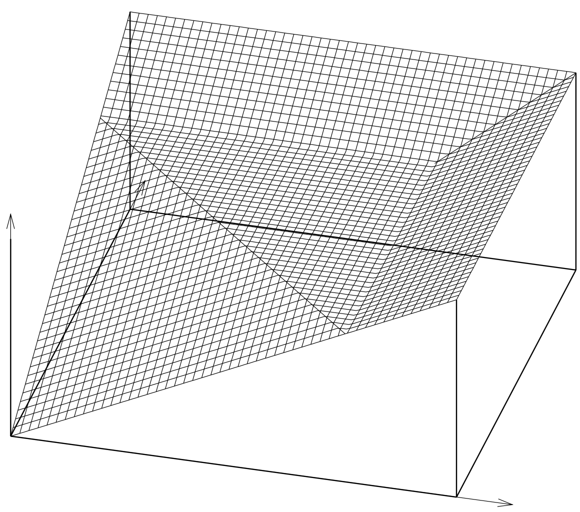


Figure 18

Note that functions of the form (*) are symmetric. The following example shows that there are Aumann functions which are not symmetric.

Example 2. Define $f : [0, +\infty)^n \to [0, +\infty)$ as follows

$$f(x) = \max\left\{ \max_{1\le i\le n}\{a_i\}, a_1 + \min\left(1, \sum_{i=2}^{n} a_i\right)\right\}.$$

It is not difficult to verify that this function is an Aumann function and $f(1, 1.5, 0, \ldots, 0) = 2 \neq 2.5 = f(1.5, 1, 0, \ldots, 0)$. (See Fig. 19. for the case $n = 2$.)

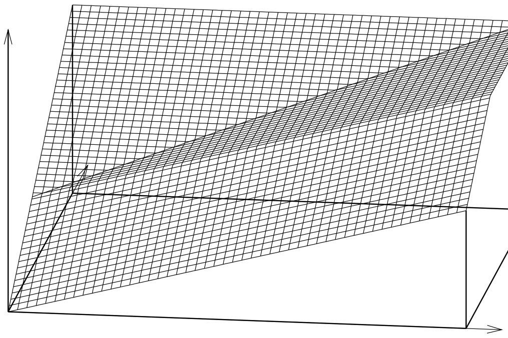


Figure 19

Denote by $\mathcal{A}_n$ the set of all Aumann functions $f : [0, +\infty)^n \to [0, +\infty)$. Then

1) $\mathcal{A}_n$ is convex, i.e. for each $p_i \in [0, +\infty)$ $(1 \le i \le n)$ with $\sum_{i=1}^{n} p_i = 1$ and for each $f_i \in \mathcal{A}_n$ $(i = 1, 2, \dots, n)$ we have $\sum_{i=1}^{n} p_i f_i \in \mathcal{A}_n$.
2) Let $f \in \mathcal{A}_n$. Let $(j_1, j_2, \dots, j_n)$ be an arbitrary permutation of $(1, 2, \dots, n)$. Define $g : [0, +\infty)^n \to [0, +\infty)$ by

$$g(x_1, x_2, \dots, x_n) = f(x_{j_1}, x_{j_2}, \dots, x_{j_n}).$$

Then $g \in \mathcal{A}_n$.
If $f \in \mathcal{A}_n$ is not symmetric, then the function $g : [0, +\infty)^n \to [0, +\infty)$ defined by

$$g(x_1, x_2, \dots, x_n) = \frac{1}{n!} \cdot \sum_{(j_1, j_2, \dots, j_n)} f(x_{j_1}, x_{j_2}, \dots, x_{j_n})$$

is a symmetric function from $\mathcal{A}_n$.

3) Let $f \in \mathcal{A}_n$, $r > 0$. Define $g : [0, +\infty)^n \to [0, +\infty)$ by

$$g(x) = \frac{1}{r} \cdot f(rx).$$

Then $g \in \mathcal{A}_n$.

14. Sums of metrics

L. Zsilinszky [68] studied properties of the sum of metrics on a given set and asked the following question:

Is it true that the sum of two metrics generating separable metric spaces generates a separable metric space again ?

The results in this section are extracted from [51].

Let X be a given nonempty set. Let d_i be an arbitrary metric on the set X for each $i = 1, 2, \dots, n$. Put

$$D = \{(x_1, x_2, \dots, x_n) \in X^n : x_1 = x_2 = \dots = x_n\}.$$

For each metric preserving function $f : [0, +\infty)^n \to [0, +\infty)$ define a metric ϱ_f on the set X as follows

$$\varrho_f(x, y) = f(d_1(x, y), d_2(x, y), \dots, d_n(x, y)).$$

It is easy to see that the metric space (X, ϱ_f) is isometric to the metric space $(D, f(d))$.

Lemma 1. *If f is a discontinuous metric preserving function, then the metric space $(D, f(d))$ is uniformly discrete.*

Proof. By Corollary 9.2 there is $\eta > 0$ such that $f(x_1, x_2, \dots, x_n) \geq \eta$ for each $x_i > 0$ $(i = 1, 2, \dots, n)$. Let $a, b \in X$, $a \neq b$. Put $x_i = d_i(a, b)$ for each $i = 1, 2, \dots, n$. Then

$$\begin{aligned} f(d((\underbrace{a, a, \dots, a}_{n}), (\underbrace{b, b, \dots, b}_{n}))) &= f(d_1(a, b), d_2(a, b), \dots, d_n(a, b)) = \\ &= f(x_1, x_2, \dots, x_n) \geq \eta. \end{aligned}$$

Now we give a positive answer to the problem of Zsilinszky. (See [51].) Suppose that (X, d_i) is separable for each $i = 1, 2, \dots, n$. If f is a continuous metric preserving function, then $(X^n, f(d))$ is a separable metric space, and $(D, f(d))$ is its separable subspace.In the case of discontinuity of f the space $(D, f(d))$ is separable iff the set D is countable.

Theorem 1. *Let (X, d_i) be a compact metric space for each $i = 1, 2, \ldots, n$. Let $f : [0, +\infty)^n \to [0, +\infty)$ be metric preserving. If f is continuous then the following conditions are equivalent:*

a) *(X, ϱ_f) is a compact space,*

b) *$(D, f(d))$ is a closed subspace of $(X^n, f(d))$.*

If f is discontinuous then (X, ϱ_f) is compact iff it is finite.

Proof. By Corollary 10.1 the metric $f(\varrho)$ generates the product topology on X^n. Hence $(X^n, f(d))$ is a compact space. Therefore the space $(D, f(d))$ is closed iff it is compact.

In the same way we can prove the following theorem.

Theorem 2. *Let (X, d_i) be a complete metric space for each $i = 1, 2, \ldots, n$. Let $f : [0, +\infty)^n \to [0, +\infty)$ be metric preserving. If f is continuous then the following conditions are equivalent:*

c) *(X, ϱ_f) is a complete space,*

d) *$(D, f(d))$ is a closed subspace of $(X^n, f(d))$.*

If f is discontinuous then (X, ϱ_f) is a complete space.

Finally we describe the relationships between the metrics $\sum_{i=1}^n d_i$ and ϱ_f.

Theorem 3. *Let $f : [0, +\infty)^n \to [0, +\infty)$ be metric preserving. Let (X, d_i) be a metric space for each $i = 1, 2, \ldots, n$. If the metric $\sum_{i=1}^n f_i$ is not discrete then the metrics ϱ_f and $\sum_{i=1}^n f_i$ are equivalent iff f is continuous.*

Proof. One part of the proof follows from Corollary 10.1. For the second part, let $(z_k)_{k \in \mathbb{N}}$ be a sequence converging to z in the metric space $(X^n, \sum_{i=1}^n d_i)$ such that $z_k \neq z$ for each $k \in \mathbb{N}$. Since ϱ_f and $\sum_{i=1}^n d_i$ are equivalent, z_k converges to z also in the metric space (X, ϱ_f), i.e.

$$\forall \varepsilon > 0 \; \exists k_0 \in \mathbb{N} \; \forall k \geq k_0 : \; d(z, z_k) < \varepsilon.$$

Put $x_i = d_i(z, z_{k_0})$ for each $i = 1, 2, \ldots, n$. This shows that

$$\forall \varepsilon > 0 \; \exists x_1, x_2, \ldots, x_n > 0 : \; f(x_1, x_2, \ldots, x_n) < \varepsilon.$$

Therefore f is continuous.

In the same way we can prove the following theorem.

Theorem 4. *Let $f : [0, +\infty)^n \to [0, +\infty)$ be metric preserving. Let (X, d_i) be a metric space for each $i = 1, 2, \ldots, n$. If the metric $\sum_{i=1}^n d_i$ is not uniformly discrete then the metrics ϱ_f and $\sum_{i=1}^n d_i$ are uniformly equivalent iff f is continuous.*

15. Modifications of the Euclidean metric on reals

Denote by $\mathcal{M}$ the set of all functions $f \in \mathcal{O}$ such that for each metric space (M, d) the function d_f is a metric on M. Denote by $\mathcal{M}_0$ ($\mathcal{M}_1$) the set of all functions $f \in \mathcal{O}$ such that e_f is a pseudometric (metric) on the real line, where $e : \mathbb{R} \times \mathbb{R} \to [0, +\infty)$ is the Euclidean metric on $\mathbb{R}$
(i.e. $e(x, y) = |x - y|$ for each $x, y \in \mathbb{R}$).

Proposition 1. *Let $f \in \mathcal{O}$. Then*

a) *$f \in \mathcal{M}_0$ iff f maps each triangle triplet $(a, b, a + b)$ to a triangle triplet;*
b) *$f \in \mathcal{M}_1$ iff $f \in \mathcal{M}_0$ and f vanishes exactly at the origin.*

Denote by F the *even extension* of $f \in \mathcal{O}$, i.e. $F : \mathbb{R} \to [0, +\infty)$, $F(x) = f(|x|)$ for each $x \in \mathbb{R}$. It is not difficult to prove

Proposition 2. *Let $f \in \mathcal{O}$. Then the following assertions are equivalent*

(i) $f \in \mathcal{M}_0$,
(ii) *F is subadditive,*
(iii) $\forall x, y \in [0, +\infty) : |f(x) - f(y)| \leq f(|x - y|)$.

Corollary 1. *Let $f \in \mathcal{M}_0$. Then f is continuous iff it is continuous at the origin.*

Proposition 3. *Let $f \in \mathcal{M}_0$. Then F is periodic with the period $t > 0$ iff $f(t) = 0$.*

Proof. Suppose that $f(t) = 0$. Let $x \in \mathbb{R}$. Then

$$F(x + t) \leq F(x) + F(t) = F(x) \leq F(x + t) + F(-t) = F(x + t),$$

which yields $F(x + t) = F(x)$.

Corollary 2. *Let $f \in \mathcal{M}_0$. Suppose that f is differentiable on the right at some $s \in f^{-1}(0)$. Then for each $t \in f^{-1}(0)$ there exist both one-sided derivatives of F and $F'_+(t) = -F'_-(t) = f'_+(s)$.*

Proposition 4. *Let $f \in \mathcal{M}_0$, $t \in F^{-1}(0)$. Then F is differentiable at t iff F is constant.*

Proof. By Corollary 2 the function F is differentiable at the origin and $F'(0) = 0$. Let $a > 0$. We shall show that $f(a) = 0$. Let $\varepsilon > 0$. Then there is $\delta > 0$ such that for each $x \in (0, \delta)$ we have

$$\frac{f(x)}{x} < \frac{\varepsilon}{a}.$$

Choose $n \in \mathbb{N}$ such that $\frac{a}{n} < \delta$. Then

$$f\left(\frac{a}{n}\right) < \frac{\varepsilon}{n}.$$

Therefore $f(a) \le n \cdot f\left(\frac{a}{n}\right) < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we obtain $f(a) = 0$.

Corollary 3. *If $f \in \mathcal{M}_0$ be nonconstant. Suppose f is differentiable on $(0, +\infty)$. Then $f \in \mathcal{M}_1$.*

The following example shows that the assumption "$f \in \mathcal{M}$" in Corollary 2.1 cannot be replacet by the assumption "$f \in \mathcal{M}_1$".

Example 1. Define $f : [0, +\infty) \to [0, +\infty)$ by

$$f(x) = |\sin x| + |\sin \sqrt{2}x|.$$

Then $f \in \mathcal{M}_1$, but $\liminf_{x \to +\infty} f(x) = 0$.

Now we will describe a construction of differentiable functions $f \in \mathcal{M}_1$ with $\liminf_{x \to +\infty} f(x) = 0$.

Lemma 1. *Let $f \in \mathcal{M}$, $n \in \mathbb{N}$. Define $f_n : [0, +\infty) \to [0, +\infty)$ as follows*

$$f_n(x) = \begin{cases} f(x) & x \in [0, 2^{n-1}], \\ f(2^n - x) & x \in (2^{n-1}, 2^n], \\ f_n(x - k.2^n) & x \in (k.2^n, (k+1).2^n], \quad (k = 1, 2, \dots). \end{cases}$$

Then $f_n \in \mathcal{M}_0$.

Proof. Let $x, y > 0$. Then there are $k, l \in \{0, 1, 2, 3, \dots\}$ such that

$$k.2^n < x \le (k+1).2^n \text{ and } l.2^n < y \le (l+1).2^n.$$

Put $a = x - k.2^n$ and $b = y - l.2^n$. Evidently $a, b \in (0, 2^n]$. Suppose that $a \le b$. Now we show that $(f_n(x), f_n(y), f_n(x+y))$ is a triangle triplet. We distinguish six cases.

1) Let $a, b, a+b \in (0, 2^{n-1}]$. Since $(a, b, a+b)$ is a triangle triplet,

$$(f_n(x), f_n(y), f_n(x+y)) = (f(a), f(b), f(a+b)) \text{ is a triangle triplet.}$$

2) Let $a, b \in (0, 2^{n-1}]$, and $a + b \in (2^{n-1}, 2^n]$. Since $(a, b, 2^n - a - b)$ is a triangle triplet,

$$(f(a), f(b), f(2^n - a - b)) \text{ is a triangle triplet.}$$

3) Let $a \in (0, 2^{n-1}]$, and $b, a+b \in (2^{n-1}, 2^n]$. Since $(a, 2^n - b, 2^n - a - b)$ is a triangle triplet,

$$(f(a), f(2^n - b), f(2^n - a - b)) \text{ is a triangle triplet.}$$

4) Let $a \in (0, 2^{n-1}]$, $b \in (2^{n-1}, 2^n]$, and $a + b \in (2^n, 3.2^{n-1}]$. Since $(a, 2^n - b, a + b - 2^n)$ is a triangle triplet,

$$(f(a), f(2^n - b), f(a + b - 2^n)) \text{ is a triangle triplet.}$$

5) Let $a, b \in (2^{n-1}, 2^n]$, and $a + b \in (2^n, 3.2^{n-1}]$. Since $(2^n - a, 2^n - b, a + b - 2^n)$ is a triangle triplet,

$$(f(2^n - a), f(2^n - b), f(a + b - 2^n)) \text{ is a triangle triplet.}$$

6) Let $a, b \in (2^{n-1}, 2^n]$, and $a + b \in (3.2^{n-1}, 2^{n+1}]$. Since $(2^n - a, 2^n - b, 2^{n+1} - a - b)$ is a triangle triplet,

$$(f(2^n - a), f(2^n - b), f(2^{n+1} - a - b)) \text{ is a triangle triplet.}$$

As a corollary we obtain

Theorem 1. *Let $f \in \mathcal{M}$. Suppose that $f(x) = 1$ for each $x \geq 1$. Define $f_0 : [0, +\infty) \longrightarrow [0, +\infty)$ as follows*

$$f_0(x) = \sup\{2^{1-n} f_n(x) : n \in \mathbb{N}\} \text{ for each } x \geq 0.$$

Then $f_0 \in \mathcal{M}_1$ and $f_0(2^n) = 2^{-n}$ for each $n \in \mathbb{N}$.

The following example shows that there is a differentiable function $f \in \mathcal{M}_1$ with $\liminf_{x \to +\infty} f(x) = 0$.

Example2. Let $f \in \mathcal{O}$ be such that (for each $n \in \mathbb{N}$)

(1) f is nondecreasing,

(2) f is differentiable on $[0, +\infty)$,

(3) $f(a_n) = 2^{1-n}$,

(4) $f'(a_n) = 0$,

(5) $f(x) = 1$ for each $x \geq 1$,

(6) $f(x) \geq k_n.x$ for each $x \in (a_{n+1}, a_n)$,

(7) $f'(x) \leq k_{n+1}$ for each $x \in (a_{n+1}, a_n)$,

where $a_n = \dfrac{n+1}{n.2^n}$ and $k_n = \dfrac{2^{1-n}}{a_n}$.

Since $f = \sup_n g_n$, where $g_n : [0, +\infty) \to [0, +\infty)$,

$$g_n(x) = \begin{cases} k_{n+1}.x & x \in [0, a_{n+1}), \\ f(x) & x \in [a_{n+1}, a_n], \\ 2^{1-n} & x \in (a_n, +\infty), \end{cases}$$

we have $f \in \mathcal{M}$. By Theorem $f_0 \in \mathcal{M}_1$ and $\liminf_{x \to +\infty} f_0(x) = 0$. It is not difficult to verify that f_0 is differentiable on $[0, +\infty)$.

By this method it is not difficult to construct a singular function $f \in \mathcal{M}_1$ with $\liminf_{x \to +\infty} f(x) = 0$.

Example 3. Let $\varphi : [0, 1] \to [0, 1]$ be the standard Cantor function.
Define $\phi : [0, +\infty) \to [0, +\infty)$ as follows

$$\phi(x) = \begin{cases} \varphi(x) & x \in [0, 1], \\ 1 & \text{otherwise.} \end{cases}$$

It is not difficult to verify that $\phi \in \mathcal{M}$. By Theorem 1 we obtain $\phi_0 \in \mathcal{M}_1$ and $\liminf_{x \to +\infty} \phi_0(x) = 0$. It is easy to see that ϕ_0 is singular.

Note that if $f \in \mathcal{M}$ is continuous, f_0 is almost periodic. In this connection a question arises of whether every continuous function $f \in \mathcal{M}_1$ with $\liminf_{x \to +\infty} f(x) = 0$ is almost periodic.

In the final part we show that the metric space $(\mathbb{R}, e_f)$ is not complete (see also [30]), where $f : [0, +\infty) \to [0, +\infty)$ is definded as follows

$$f(x) = \begin{cases} x, & \text{if } x \in [0,1], \\ \max(2^{-n}, |x-m|), & \text{if } x \in [m-1, m+1], \end{cases}$$

where $m = 2^n(2k-1)$ (for each $k, n \in \mathbb{N}$). It is not difficult to verify that $f \in \mathcal{M}_1$ and $f(2^n) = 2^{-n}$.

For each $n \in \mathbb{N}$ put

$$u_n = \frac{10 \cdot 4^n - 1}{3}.$$

Evidently each u_n is an odd natural number. Therefore $f(u_n) = 1$ for each $n \in \mathbb{N}$. It is easy to see that the sequence $(u_n)_{n\in\mathbb{N}}$ is increasing and $u_n \xrightarrow{e} +\infty$.

Now we show that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the metric space $(\mathbb{R}, e_f)$. Suppose $k, n \in \mathbb{N}$. Then

$$u_{n+k} - u_n = \frac{10 \cdot 4^{n+k} - 1}{3} - \frac{10 \cdot 4^n - 1}{3} = 4^n \cdot 10 \cdot \frac{4^k - 1}{3}.$$

Since $\dfrac{4^k - 1}{3}$ is an odd natural number, we obtain

$$f(|u_{n+k} - u_n|) = \frac{1}{2 \cdot 4^n}.$$

Now we show that $(u_n)_{n\in\mathbb{N}}$ is not convergent in the metric space $(\mathbb{R}, e_f)$. By contradiction. Suppose that $u_n \xrightarrow{e_f} u$. First we show that u is an odd integer number. Since

$$|f(u_n) - f(|u|)| \leq \underbrace{f(|u_n - u|)}_{\xrightarrow{e} 0},$$

we have $1 = f(u_n) \xrightarrow{e} f(|u|)$. Thus $|u|$ is an odd natural number.

Finally, we show that for each $n \in \mathbb{N}$, $2^n \geq |3u + 1|$,

$$f(|u_n - u|) \geq \frac{1}{|3u + 1|}.$$

Let $r > 0$ and s be integer numbers such that $3u + 1 = 2^r(2s - 1)$. Since $|3u + 1| = 2^r \cdot |2s - 1| \geq 2^r$, we obtain $2^n \geq 2^r$.

$$\text{Since } u = \frac{2^r(2s-1)-1}{3}, \text{ we have}$$

$$u_n - u = 2^r \cdot \underbrace{\frac{2^{2n+1-r} \cdot 5 - 2s + 1}{3}}_{\text{odd integer}}, \text{ thus}$$

$$f(|u_n - u|) = 2^{-r} \geq \frac{1}{|3u+1|}, \text{ a contradiction.}$$

Bibliography

[1] Aumann, G., *Reelle Funktionen*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1954.

[2] Berz, E., *Sublinear functions on* $\mathbb{R}$, Aeq. Math. **12** (1975), 200–206.

[3] Billingsley, P., *Van der Waerden's continuous nowhere differentiable function*, Amer. Math. Montly **89** (1982), p. 691.

[4] Borsík, J. and Doboš, J., *Functions whose composition with every metric is a metric*, Math. Slovaca **31** (1981), 3–12. (in Russian)

[5] Borsík, J. and Doboš, J., *On a product of metric spaces*, Math. Slovaca **31** (1981), 193–205.

[6] Borsík, J. and Doboš, J., *On metrization of the uniformity of a product of metric spaces*, Math. Slovaca **32** (1982), 97–102.

[7] Borsík, J. and Doboš, J., *On metric preserving functions*, Real Anal. Exch. **13** (1987-88), 285–293.

[8] Bourbaki, N., *Topologie générale*, (Russian translation, Nauka, Moscow, 1968).

[9] Bruckner, A., *Minimal superadditive extensions of superadditive functions*, Pacific J. Math. **10** (1960), 1155–1162.

[10] Bruckner, A., *Differentiation of Real Functions*, Lecture Notes in Mathematics 659, Springer-Verlag, New York, 1978.

[11] Burdyuk, V.J., *On metric preserving functions*, VINITI No. 6335-V90, Univ. of Dnepropetrovsk (1990). (in Russian)

[12] Burdyuk, V.J., *Distances*, Univ. of Dnepropetrovsk, 1993, ISBN 5-86400-235-6. (in Ukrainian)

[13] Charatonik, J. J. and Charatonik, W. J., *The set of additivity of the Cantor-Lebesgue step function*, (submitted).

[14] Cooper, R., *The converses of the Cauchy-Hölder inequality and the solutions of the inequality* $g(x+y) \leq g(x) + g(y)$, Proc. London Math. Soc. **26** (1927), 415–432.

[15] Corazza, P., *Introduction to metric-preserving functions*, Amer. Math. Monthly, (to appear).

[16] Darst, R.B., *Some Cantor sets and Cantor functions*, Math. Mag. **45** (1972), 2–7.

[17] Das, P. P., *Metricity preserving transforms*, Pattern Recognition Letters **10** (1989), 73–76.

[18] Doboš, J., *On a certain lattice of topologies on a product of metric spaces*, Math. Slovaca **32** (1982), 397–402.

[19] Doboš, J., *The standard Cantor function is subadditive*, Proc. Amer. Math. Soc. **124** (1996), 3425–3426.

[20] Doboš, J., *On modifications of the Euclidean metric on reals*, Tatra Mt. Math. Publ. **8** (1996), 51–54.

[21] Doboš, J., *A survey on metric preserving functions*, Q & A in General Topology **13** (1995), 129–134.

[22] Doboš, J. and Piotrowski, Z., *Some remarks on metric preserving functions*, Real Anal. Exch. **19** (1993-94), 317–320.

[23] Doboš, J. and Piotrowski, Z., *A note on metric preserving functions*, Internat. J. Math. & Math. Sci. **19** (1996), 199–200.

[24] Doboš, J. and Piotrowski, Z., *When distance means money*, Int. J. Math. Educ. Sci. Technol. **28** (1997), 513–518.

[25] Gaskill, H. S. and Narayanaswami, P. P., *Foundations of analysis. The theory of limits*, Harper & Row Publishers, New York, 1989.

[26] Ger, R. and Kuczma, M., *On the boundedness and continuity of convex functions and additive functions*, Aeq. Math. **4** (1970), 157–162.

[27] de Groot, J., *Non-Archimedean metrics in topology*, Proc. Amer. Math. Soc. **7** (1956), 948–953.

[28] Guričan, J., *Strengthening of the triangle inequality*, Tatra Mt. Math. Publ. **8** (1996), 211–216.

[29] Herburt, I. and Moszyńska, M., *On metric products*, Colloq. Math. **62** (1991), 121–133.

[30] Hattori, Y., *A metrizable group topology on the real line with special convergences*, Math. Slovaca, (to appear).

[31] Hille, E. and Phillips, R. S., *Functional Analysis and Semigroups*, AMS Coll. Publ., Vol. 31, Providence, RI, 1957.

[32] Jůza, M., *A note on complete metric spaces*, Matematicko-fyzikálny časopis SAV **6** (1956), 143–148. (in Czech)

[33] Kelley, J. L., *General Topology*, Van Nostrand, New York, 1955.

[34] Kuhn, N., *A note on t-convex functions*, General Inequalities 4, International Series of Numerical Mathematics, Vol. 71, Birkhäuser Verlag, Basel (1984), 269–276.

[35] Lassak, M., *On Helly's dimension of products of metric spaces*, Mat. issled., Kishinev **X, 2 (36)** (1975), 159–167.

[36] Matkowski, J., *On a characterization of L^p-norm*, Annales Polon. Math. **50** (1989), 137–144.

[37] Matkowski, J., *Subadditive functions and relaxation of the homogenity condition of seminorms*, Proc. Amer. Math. Soc. **117** (1993), 991–1001.

[38] Matkowski, J., *The converse of the Minkowski's inequality theorem and its generalization*, Proc. Amer. Math. Soc. **109** (1990), 663–675.

[39] Matkowski, J. and Świątkowski, T., *On subadditive functions*, Proc. Amer. Math. Soc. **119** (1993), 187–197.

[40] Matkowski, J. and Świątkowski, T., *Quasi-monotonicity, subadditive bijections of $\mathbb{R}_+$, and characterization of L^p-norm*, J. Math. Anal. and Appl. **154** (1991), 493–506.

[41] Matkowski, J. and Świątkowski, T., *Subadditive functions and partial converses of Minkowski's and Mulholland's inequalities*, Fund. Math. **143** (1993), 75–85.

[42] Moreau, J.-J., *Inf-convolution, sous-additivité, convexité des fonctions numérique*, J. Math. Pures et Appl. **49** (1970), 109–154.

[43] Moszner, Z., *Sur une hypothèse au sujet des functions subadditives*, Aeq. Math. **2** (1969), 380–386.

[44] Motzkin, Th., *Sur le produit des espaces métriques*, Congrès International des Mathématiciens (Oslo, 1936) **II**, 137–138.

[45] Mulholland, H. P., *On generalizations of Minkowski's inequality in the form of a triangle inequality*, Proc. London Math. Soc. **51** (1950), 294–307.

[46] Neubrunn, T. and Šalát, T., *Über eine Klasse metrischer Räume*, Acta F.R.N. Univ. Comen. X, 3, Math. XII. (1965), 23–30.

[47] Olędzki, J. and Spież, S., *Remarks on intrinsic isometries*, Fund. Math. **119** (1983), 241–247.

[48] Piotrowski, Z., *On integer-valued metrics*, School of Mathematics, Physics and Chemistry, Wrocław University, Poland, 1974. (in Polish)

[49] Pokorný, I., *Some remarks on metric preserving functions*, Tatra Mt. Math. Publ. **2** (1993), 65–68.

[50] Pokorný, I., *Some remarks on metric preserving functions of several variables*, Tatra Mt. Math. Publ. **8** (1996), 89–92.

[51] Pokorný, I., *Remarks on the sum of metrics*, Tatra Mt. Math. Publ. **14** (1998), 63–65.

[52] Richey, M., *Mapping the Cantor set onto [0,1] : a new construction*, preprint.

[53] Rosenbaum, R. A., *Subadditive functions*, Duke Math. J. **17** (1950), 227–247.

[54] Šalát, T., Tóth, J. T. and Zsilinsky, L., *Metric space of metrics defined on a given set*, Real Anal. Exch. **18** (1992–93), 225–231.

[55] Šalát, T., Tóth, J. T. and Zsilinsky, L., *On the structure of the space of metrics defined on a given set*, Real Anal. Exch. **19** (1993–94), 321–327.

[56] Shirai, T., *On the relations between the set and its distances*, Mem. Coll. Sci. Kyoto Imp. Univ. **Ser. A 22** (1939), 369–375.

[57] Sierpiński, W., *Sur un système d'équations fonctionnelles définissant une fonction avec un ensemble dense d'intervalles d'invariabilité*, Bulletin International de l'Académie des Sciences et des Lettres de Cracovie, Cl. Sci. Math. Nat. Sér. A (1911), 577–582.

[58] Sreenivasan, T. K., *Some properties of distance functions*, J. Indian Math. Soc. (N.S.) **11** (1947), 38–43.

[59] Strömberg, T., *An introduction to the operation of infimal convolution*, Research report, Dept. of Math., Luleå University, 1994.

[60] Terpe, F., *Metric preserving functions*, Proc. Conf. Topology and Measure IV, Greifswald (1984), 189–197.

[61] Terpe, F., *Metric preserving functions of several variables*, Proc. Conf. Topology and Measure V, Greifswald (1988), 169–174.

[62] Terpe, F., *Some properties of metric preserving functions*, Proc. Conf. Topology, Measure and Fractals, Math. Res. 66, Akademie-Verlag, Berlin (1992), 214–217.

[63] Toranzos, F., *Sobre la inecuación funcional* $f(x) + f(y) \geqq f(x + y)$, Rev. Mat. Hisp.-Amer. **8** (1933), 109–113.

[64] Vallin, R. W., *A subset of metric preserving functions*, Internat. J. Math. & Math. Sci., (to appear).

[65] Vallin, R. W., *On metric preserving functions and infinite derivatives*, Acta Math. Univ. Comenian. (N.S.), (to appear).

[66] Vallin, R. W., *Metric preserving functions and differentiation*, Real Anal. Exch. **22** (1996-97), p. 86.

[67] Wilson, W. A., *On certain types of continuous transformations of metric spaces*, Amer. J. Math. **57** (1935), 62–68.

[68] Zsilinsky, L., *On the sum of metrics generating metric spaces of prescribed properties*, Acta Math. F. N. S. Univ. Edu. Nitra **2** (1995), 93–96.

Dátum vyhotovenia: október 1998
Počet výtlačkov: 65
Vydalo: vydavateľstvo Štroffek, Košice, Slovakia
ISBN: 80-88896-30-4