Problem B-4. Prove that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive real numbers, then so is $\sum_{n=1}^{\infty} (a_n)^{n/(n+1)}$.

Solution I.
`http://www.math.hawaii.edu/~dale/putnam/1988.pdf`

Let $S = \{n : (a_n)^{n/(n+1)} < 2a_n\}$.
If $n \notin S$, $(a_n)^{n/(n+1)} \geq 2a_n$, or equivalently $1/2 \geq (a_n)^{1-n/(n+1)} = (a_n)^{1/(n+1)}$, which is the same as $1/2^n \geq (a_n)^{n/(n+1)}$. It follows that

$$\sum_{n=1}^{\infty} (a_n)^{n/(n+1)} \leq \sum_{n \in S} (a_n)^{n/(n+1)} + \sum_{n \notin S} 1/2^n < \infty.$$

Solution II. (by Jozef Doboš, P J Šafárik University, Košice)

By the Arithmetic and Geometric Mean Inequality,

$$a_n^{\frac{n}{n+1}} = \sqrt[n+1]{\frac{1}{n+3} \cdot (n+3)a_n \cdot a_n^{n-1}} \leq$$

$$\leq \frac{\frac{1}{n+3} + (n+3)a_n + (n-1)a_n}{n+1} = \frac{1}{(n+1)(n+3)} + 2a_n.$$

Problem No. 8. Assume that $a_n > 0$ for each n, and that

$$\sum_{n=1}^{\infty} a_n$$

converges. Prove that

$$\sum_{n=1}^{\infty} a_n^{\frac{n-1}{n}}$$

converges as well.

Solution I. (by Georges Ghosn, Quebec)

`http://www.math.purdue.edu/pow/fall2005/pdf/solution8.pdf`

We have for $n \geq 2$,

$$a_n^{\frac{n-1}{n}} = (a_n^{1/2} a_n^{1/2} \cdot a_n^{n-2})^{\frac{1}{n}} \leq \frac{2\sqrt{a_n} + (n-2)a_n}{n} \quad \text{(Arithmetic-geometric Inequality)}$$

But $\dfrac{2\sqrt{a_n}}{n} \leq \dfrac{1}{n^2} + a_n$ (because $2xy \leq x^2 + y^2$),

and $\dfrac{(n-2)a_n}{n} \leq a_n$ (because $\dfrac{n-2}{n} \leq 1$).

Therefore, $0 < a_n^{\frac{n-1}{n}} \leq \dfrac{1}{n^2} + 2a_n$, for each $n \geq 1$. Finally the comparison test shows that $\sum_{n=1}^{\infty} a_n^{\frac{n-1}{n}}$ converges since $\sum_{n=1}^{\infty} \left(\dfrac{1}{n^2} + 2a_n \right) = \sum_{n=1}^{\infty} \dfrac{1}{n^2} + 2 \sum_{n=1}^{\infty} a_n$ clearly converges.

Solution II. (by the Panel)

`http://www.math.purdue.edu/pow/fall2005/pdf/solution8.pdf`

Each term a_n satisfies either the inequality $0 < a_n \leq \dfrac{1}{2^n}$ or $\dfrac{1}{2^n} < a_n$. In the first case, $a_n^{\frac{n-1}{n}} \leq \dfrac{1}{2^{n-1}}$. In the second one, $a_n^{\frac{n-1}{n}} = \dfrac{a_n}{a_n^{\frac{1}{n}}} \leq 2a_n$. Therefore, in both cases,

$$0 < a_n^{\frac{n-1}{n}} \leq \frac{1}{2^n} + 2a_n.$$

The conclusion is now immediate since $\sum_{n=1}^{\infty} \dfrac{1}{2^n}$ converges, and so does $\sum_{n=1}^{\infty} 2a_n$.