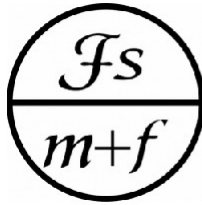


GEYSER MATHEMATICAE CASSOVIENSIS



Košice
August 2019



Jednota
slovenských
matematikov
a fyzikov
pobočka Košice



Slovenská spoločnosť aplikovanej kybernetiky
a informatiky pri ÚRIVP Fakulty BERG TUKE

GEYSER MATHEMATICAE CASSOVIENSIS



Košice
2019

GEYSER MATHEMATICAE CASSOVIENSIS

Erika Fecková Škrabuľáková
(Eds.)

Cover design by: Erika Fecková Škrabuľáková
Published by: Technical University of Košice, Košice, Slovakia

All rights reserved
© 2019

ISBN 978 - 80 - 553 - 3327 - 4

This work was supported by the Slovak Research and Development Agency under the contract No. APVV-14-0892. This work was also supported by the Union of Slovak Mathematicians and Physicists, division Košice (JSMF), SSAKI by URIVP FBERG, SAV and ZSVTS.

After completing the double blind peer reviewed process of the collection of papers Geysler Mathematicae Cassoviensis the acceptance rate was 80 %.

Introduction to GMC

Dear reader, you are opening a peer review publication dedicated to new trends in Košice's mathematics - **Geyser Mathematicae Cassoviensis (GMC)** with contributions soundly based in research or scholarship. It seeks to cover the whole field of post-school mathematical education and/or research in all areas of mathematics. It aims to take a problem-oriented approach; to help formulate the problems of higher education, to consider alternative solutions and to test them. Lastly, it seeks to inform about the up to date research in Košice and Košice's surround via research papers, review articles and/or short communications.

The education reveals abilities but do not create them. Universities educate new generations that form the national elite. Well educated people become new workers and open-minded researchers. This points out to the fact that educational activities and problems have their important place in nowadays scientific discussions.

A successful academic career is increasingly linked to a track record of publishing research which is able to reach a large audience. Geyser Mathematicae Cassoviensis aims to help especially the young researchers and PhD. students in the field. Every impact research increasingly underpins their future research grants and career progression.

This publication is addressed to all those who work with or within the field of mathematics: students, academics, administrators and policymakers at all levels. Via this publication we want to faster the research in matematics. We hope that the obtained information encourage you in your study, education, teaching or research activities.

The publication adoped a double blind reviewing policy - both the referee and author remained anonymous to each other during the review process. It was mandatory that all co-authors have seen and approved the submitted version of the paper and that manuscripts submitted to Geyser Mathematicae Cassoviensis have not been published in any other journal before.

Authors are encouraged to self-archive their manuscripts and enable public access from their institution's repository. This work can be used under the "Attribution Non-Comercial Share Alike" licence for educational purposes.

Erika Fecková Škrabuláková

Using GeoGebra for solving equations and inequalities

Jozef Doboš¹

Abstract: In this article, we show how to use GeoGebra for solving equations and inequalities in school Mathematics. Many solutions offered by GeoGebra in its CAS View are not too dissimilar from the answers expected in a school context. However, there are multiple differences that could confuse both students and teachers. Using actual worked examples, the aim of this article is to point out some of the reasons for these differences and to demonstrate how even in such situations GeoGebra can deliver solutions expected in school Mathematics.

Keywords: GeoGebra; Equations; Inequalities.

Mathematics Subject Classification: 97U70.

1 Introduction

The GeoGebra software is a powerful tool for solving equations and inequalities in school Mathematics. Many solutions offered by this tool are similar to the answers expected in a school context. However, there are multiple differences that could confuse both students and teachers. The aim of this article is to highlight several reasons for these discrepancies. We also show how even in such situations solutions expected in school Mathematics can be delivered via GeoGebra.

2 Solving some equations and inequalities by a software

Equations and inequalities are to be solved in the computer algebra window (CAS View). In order to effectively use this tool, students and teachers need to understand the sources of unexpected solutions and have an idea how to deal with them.

¹Institute of Mathematics, Faculty of Science, P J Šafárik University, Jesenná 5, 040 01 Košice, Slovakia, e-mail: jozef.dobos@upjs.sk

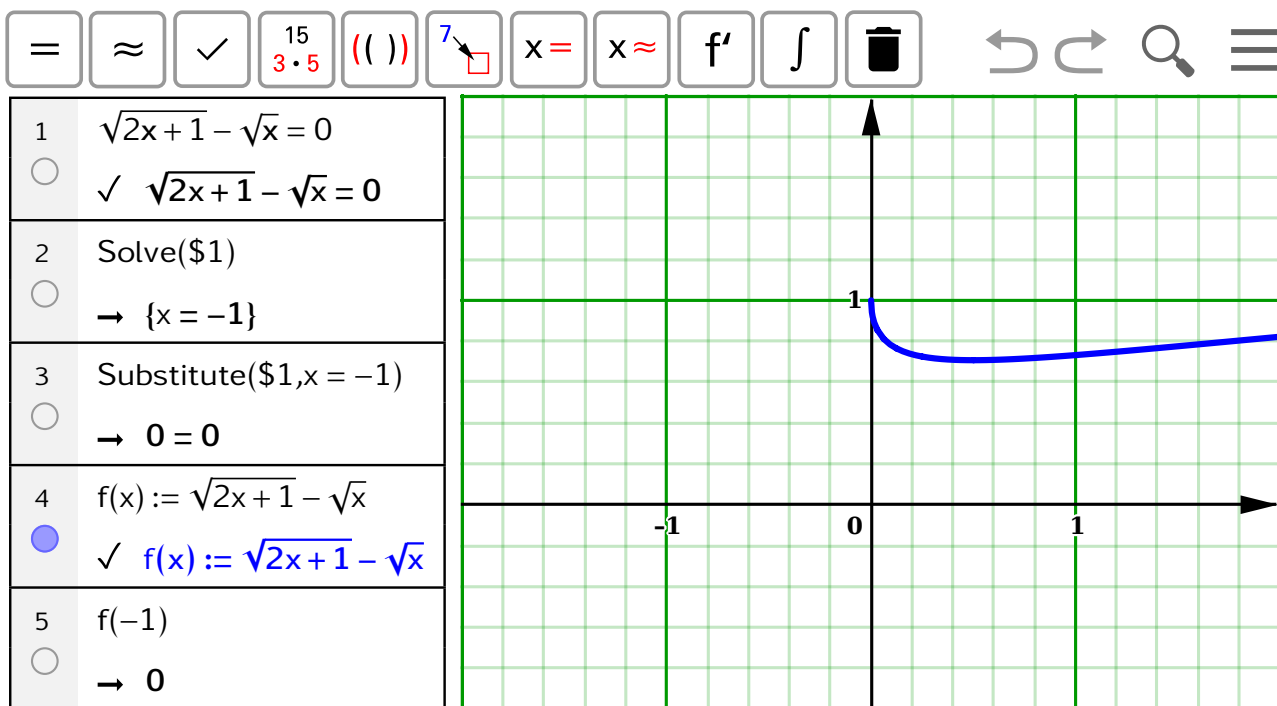


Figure 1: Solving the equation $\sqrt{2x+1} - \sqrt{x} = 0$

The first example is a solution to equation

$$\sqrt{2x+1} - \sqrt{x} = 0. \quad (1)$$

Put equation (1) into cell No. 1 either using a virtual keyboard or by directly typing `sqrt(2x+1)-sqrt(x)=0` in there (see Figure 1). Next, using a mouse, click the button (third left). This ensures the insertion of an expressions without (not always correct) changes.

Write `Solve($1)` into cell No. 2, where `Solve(<equation>)` is a command for solving equations, and `$1` is a reference to the contents of a cell No. 1. Afterwards, click the button in the top toolbar (first left). GeoGebra will print out a list of solutions to this equation, hence the angle brackets. GeoGebra is trying to persuade us that equation (1) has exactly one real root $x = -1$.

To check this solution, we can substitute it back into the equation saved in a cell No. 1. Thus, write `Substitute($1,x=-1)` into cell No. 3. Next, click the button in the top toolbar (first left). The result is $0 = 0$, verifying that $x = -1$ is a valid solution (by GeoGebra). However, we know that $x = -1$ cannot be a solution to (1) since negative numbers cannot appear under a square root sign. This is what students are thought in secondary schools. Something not being correct is even shown by GeoGebra itself in the graphical window. Write `f(x):=sqrt(2x+1)-sqrt(x)` in a cell No. 4. That is how in CAS window one defines a function f using $f(x) = \sqrt{2x+1} - \sqrt{x}$. Next, click the button in the top toolbar (third left). GeoGebra will plot a

graph of a function f , what is actually the left-hand side of equation (1).

Write $f(-1)$ in a cell No. 5 in order to check whether GeoGebra can calculate the value of function f at $x = -1$. Next, click the \square button in the top toolbar (first left). GeoGebra is trying to persuade us that $f(-1) = 0$. However, that would mean that the point $(-1, 0)$ lies on a graph of the function f , but it does not. There is clearly a discrepancy.

An explanation is found when we solve the equation

$$\sqrt{2x+1} = \sqrt{x} \quad (2)$$

instead of equation (1).

Actually, equation (1) came from equation (2) which can be found in [5]. That is a dissertation which deals on solving equations in school Mathematics using computer algebra systems (CAS). The substitution in a cell No. 3 shows that after plugging $x = -1$ into equation (2), equality $\iota = \iota$ is obtained, where ι is a special symbol used by GeoGebra for imaginary values. This must not be confused with the letter i . However, we can insert it directly using keyboard shortcut $\text{Alt}+i$, or via a virtual keyboard. See Figure 2.

1	$\sqrt{2x+1} = \sqrt{x}$
<input type="radio"/>	$\checkmark \sqrt{2x+1} = \sqrt{x}$
2	Solve(\$1)
<input type="radio"/>	$\rightarrow \{x = -1\}$
3	Substitute(\$1, x = -1)
<input type="radio"/>	$\rightarrow \iota = \iota$

Figure 2: Solving the equation $\sqrt{2x+1} = \sqrt{x}$

This way of solving irrational equations is not specific to GeoGebra. For instance, commercial software Maple behaves in the exact same manner. The only difference is in the labelling of the imaginary values as can be seen in an example on Figure 3. CAS systems usually calculate the principal root (главное значение корня), cf. [4] and [1]. If non-zero complex number z is expressed in a trigonometric form $z = r(\cos \theta + i \sin \theta)$, where $-\pi < \theta \leq \pi$, then the principal value of n -root of z is a complex number $w_0 = \sqrt[n]{r} \left(\cos \frac{\theta}{n} + i \sin \frac{\theta}{n} \right)$.

$$\begin{array}{l}
\text{solve}(\sqrt{2x+1} - \sqrt{x} = 0) \\
-1 \\
f := x \rightarrow \sqrt{2x+1} \\
x \rightarrow \sqrt{2x+1} \\
f(-1) \\
I \\
g := x \rightarrow \sqrt{x} \\
x \rightarrow \sqrt{x} \\
g(-1) \\
I
\end{array}$$

Figure 3: Solving the equation $\sqrt{2x+1} - \sqrt{x} = 0$ via Maple

For GeoGebra to solve equation (2), as would be expected in school Mathematics, the domain of the equation has to be taken into account, too. That can be obtained by solving a system of inequalities $2x + 1 \geq 0$, $x \geq 0$. It is sufficient in a cell No. 1 in the CAS window to type

$$\text{Solve}(\{\text{sqrt}(2x+1)=\text{sqrt}(x), 2x+1 \geq 0, x \geq 0\})$$

By doing so, we solve a mixed system; one equation and two inequalities (that define a domain of the equation). Next, click \square button in the top toolbar (first right). GeoGebra now prints out a correct result – an empty list in a form of $\{\}$.

We can also try the suggested procedure on the following equation

$$\sqrt{2x^2 + 5x + 1} = \sqrt{x + 1}. \quad (3)$$

Command $\text{Solve}(\text{sqrt}(2x^2+5x+1)=\text{sqrt}(x+1))$ results in a list of solutions $\{x=-2, x=0\}$, out of which $x = -2$ does not belong to the domain of equation (3). However, command

$$\text{Solve}(\{\text{sqrt}(2x^2+5x+1)=\text{sqrt}(x+1), 2x^2+5x+1 \geq 0, x+1 \geq 0\})$$

results in an expected list of solutions $\{x=0\}$.

Interesting is also the situation with a cube root. Our secondary school textbooks explicitly say that we do not define n -th root of negative numbers. See for instance [3]. Only in textbook [2], it is mentioned that it is possible to define a cube root of any (even negative) number. But immediately in the next sentence it is said that we are only interested in roots of positive numbers and indeed it is being followed. Therefore, when solving equation

$$\sqrt[3]{24+x} + \sqrt{12-x} = 6 \quad (4)$$

one must first decide whether to allow for a cube root of negative numbers or not. GeoGebra will otherwise include also number $x = -88$ among the solutions.

Similarly to the above, in program GeoGebra, one is solving a mixed system consisting of an equation and one or two inequalities determining the domain of that equation.

The difference between buttons \checkmark and $=$ can be tested, for instance, using equation

$$\sqrt{x-1}\sqrt{x-2} = 1. \quad (5)$$

GeoGebra struggles with equations that have infinitely many solutions. As an example, try to solve the equation

$$\sqrt{x-1} = 1 + \sqrt{x-2\sqrt{x-1}}. \quad (6)$$

A solution of equation (6) is any real number $x \geq 2$. This can be verified via a graph of the function $f(x) = 1 + \sqrt{x-2\sqrt{x-1}} - \sqrt{x-1}$. See Figure 4.

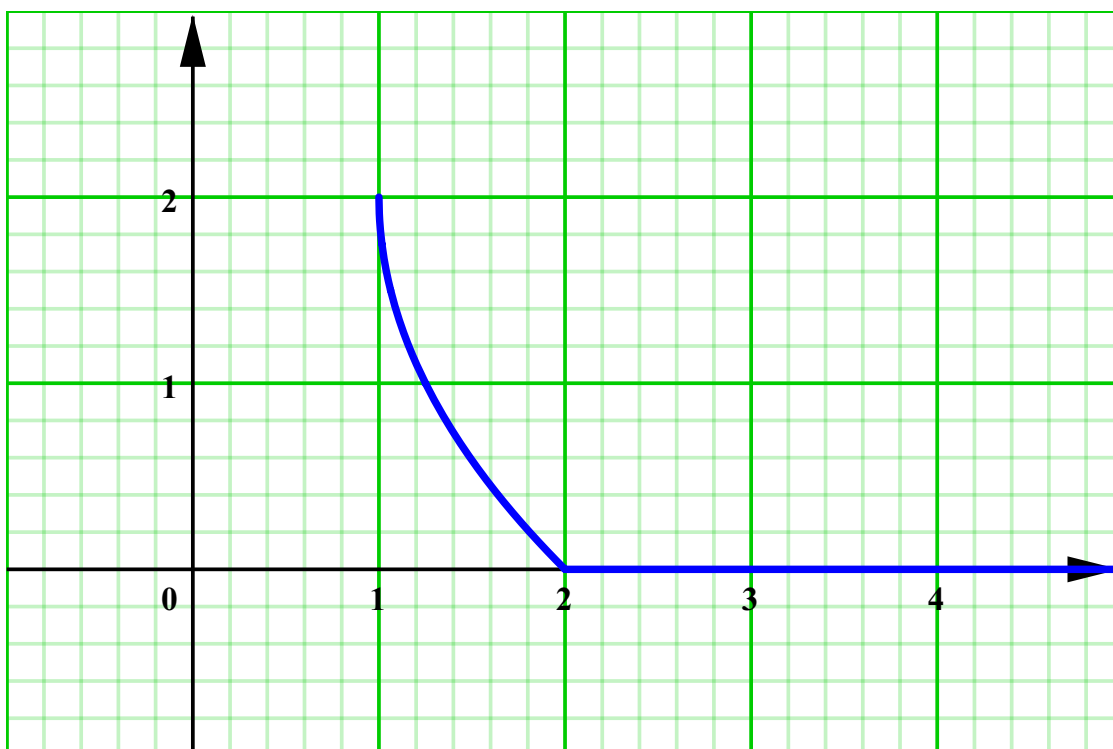


Figure 4: The graph of the function $f(x) = 1 + \sqrt{x-2\sqrt{x-1}} - \sqrt{x-1}$

One needs to be careful also when solving equations with parameters since GeoGebra does not offer a discussion regarding the parameters. As an example, try to solve equation

$$\frac{1}{x+a+b} = \frac{1}{x} + \frac{1}{a} + \frac{1}{b}. \quad (7)$$

GeoGebra returns roots $x = -a$, $x = -b$. Even when checking solutions, it attempts to persuade us that it is correct. However, it is sufficient to move everything to the right-hand side and then factoring it, for example use command

`Factor(0=1/x+1/a+1/b-1/(x+a+b))`. After this arrangement, a discussion regarding the parameters can be achieved relatively easily.

When dealing with logarithmic equations and inequalities, it is also recommended to solve mixed systems, similarly to how it was done with irrational equations. Try to solve equation

$$\log_{10}(x - 2) - \log_{10}(x - 1) = 1. \quad (8)$$

Sometimes it is needed to help oneself with a command `Simplify`. For instance, when solving inequality

$$\left(\frac{1}{2}\right)^{\log_2(x^2-1)} < 1 \quad (9)$$

using command `Solve({Simplify((1/2)^(ld(x^2-1))<1),x^2-1>0},x)` one gets $\{-\sqrt{2} > x, x > \sqrt{2}\}$. That is the list of intervals the union of which defines the solution set for inequality (9).

A problem with domain also appears when solving trigonometric equations, for instance

$$\frac{1 - \cos(2x)}{\sin(2x)} = 0. \quad (10)$$

Equation (10) is solved using command `Solve({(1-cos(2x))/sin(2x)=0,sin(2x)!=0},x)`.

Even bigger problem GeoGebra encounters with equation

$$\sqrt{\cos x - 1} = 0. \quad (11)$$

This is because the domain of equation (11) consist of isolated points. Equation (11) is solved using command `Solutions(sqrt(cos(x)-1)=0)`.

3 Conclusion

GeoGebra is not perfect and that is fine. It will not do everything instead of us. To get the correct answers, one must first know something about Mathematics and also needs to know how to use it.

Acknowledgement

This work was supported by the Slovak Scientific Grant Agency VEGA under the contract No. 1/0265/17.

Bibliography

- [1] И. И. Бронштейн, К. А. Семендяев: *Справочник по математике для инженеров и учащихся втузов*, «Наука», Москва, 1986. (In Russian)
- [2] Z. Kubáček: *Matematika pre 2. ročník gymnázií a 6. ročník gymnázií s osmročným štúdiom*, druhá časť, Orbis Pictus Istropolitana, Bratislava, 2010. (In Slovak)
- [3] O. Odvárko, M. Božek, M. Ryšánková, J. Smida: *Matematika pre 2. ročník gymnázia*, SPN, Bratislava, 1985. (In Slovak)
- [4] И. И. Яцкин: *Алгебра. Теоремы и алгоритмы, учеб. пособие*, Иван. гос. ун-т, Иваново, 2006. (In Russian)
- [5] E. Tõnisson: *Differences Between Expected Answers and the Answers Offered by Computer Algebra Systems to School Mathematics Equations*, Dissertationes Mathematicae Universitatis Tartuensis 122, University of Tartu Press, 2017.

Article history:

Received: 2019/05/26

Received in revised form: 2019/06/17

Accepted: 2019/06/19

RNDr. Erika Fecková Škrabuľáková, PhD. (Eds.)

GEYSER MATHEMATICAE CASSOVIENSIS

Technical University of Košice
Košice, Slovakia

Impression: 50

Edition: first

Pages: 73

Košice

2019

ISBN 978 - 80 - 553 - 3327 - 4