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On the existence of equitable cake divisions*

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Abstract

Let the cake be represented by the unit interval and let each player have a valuations expressed by a nonatomic probability measure. A cake division is said to be equitable if the value of a piece assigned to a player by her measure is the same for all players. We show that for any number n of players an equitable division exists, giving each player a contiguous cake piece.

Keywords. Cake cutting, equitable division.

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1 Introduction

In this paper we deal with the problem of 'fairly' dividing a certain infinitely divisible resource, traditionally called the cake, between n people (players). The cake is represented by the interval $[0, 1]$ of reals. Players may have different opinions about the values of different parts of the cake.

We shall concentrate on equitable divisions, i.e. such that the values of pieces assigned to all players are equal (according to their valuations). In the literature, other concepts of fairness are considered, too. In a proportional division each player receives at least $1/n$ part of the cake according to her valuation, in an envy-free division no player thinks that she would be better off with somebody else's piece and an exact division assigns pieces such that if a player evaluates by her own measure a piece assigned to any player (including herself), she always gets exactly $1/n$. It is known that in general, these properties are not equivalent, see e.g. [3] and [9], where also some other notions are defined and the relations

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between them explored. For example, an exact division is always equitable, but an equitable division may be neither proportional nor envy-free. It might even happen that an equitable division gives to each player a piece of value zero, so this paradox undermines our feeling of fairnes. This might also be a possible explanation why equitability is not as popular as proportionality or envy-freeness criteria.

A result of Alon [1] implies that an exact division for n players always exists, however, as many as $(n-1)n$ cuts may be necessary. Equitable divisions with contiguous pieces have also been studied. Brams, Jones and Klamler [3] assumed that the valuations are in the form of an integral of a Riemann integrable function (probability densities). The cutpoints of an equitable division for n players having $n-1$ points are then solutions of a system of equations involving integrals, where the unknowns are the upper and/or lower bounds of the integrals. However, they did not prove the solvability of such a system. Mawet, Pereira and Petit [8] considered the case of piecewise constant density functions. They assumed that their breakpoints, dividing $[0,1]$ into subintervals with constant nonzero densities, as well as the values of these constants are known. Under such assumptions, given the order of players and for each cutpoint the subinterval where it is located, the precise positions of the cutpoints are a solution of a system of linear equations. Mawet, Pereira and Petit showed that the matrix of this system is always regular, hence a unique solution exists, however, they did not show that this solution really falls into subintervals where they assumed it is located. Another existence proof for equitable divisions is given in [2]. It uses the compactness of the set of all divisions with contiguous pieces. Our result is stronger in that we prove the existence of such divisions for any order of players.

2 Definitions

We will consider the set of *players* $N = \{1, 2, ..., n\}$. The *cake* is represented by the interval $[0,1]$. In this work, the only *allowable portions – pieces* (see [7]) are intervals $[p,q]$, $0 \le p \le q \le 1$. A cutpoint of two neighbouring pieces cannot belong to both of them, but since in our model the value of a piece is not influenced by a single point, we shall represent all pieces as closed intervals.

We shall suppose that each player i is endowed with a nonatomic probability measure U_i on the cake. Such a measure can be represented by the distribution function $F_i(x) = U_i(0,x)$, so that the measure of each interval $[p,q]$ is equal to $F_i(q) - F_i(p)$. The properties of the measure imply that the function F_i is nonnegative, nondecreasing and continuous on [0,1] and $F_i(0) = 0, F_i(1) = 1$. If the distribution function F_i has a density f_i, then

$$U_i(p,q) = \int_p^q f_i(t)dt.$$

A cake division is a partition of the cake into n disjoint pieces; the piece assigned to player i in a division D will be denoted by D_i. The various fairness criteria are formulated in the following definition (see also [3, 9] for other notions and relations between them).

Definition 1 *A cake division $D = (D_1, D_2, \dots, D_n)$ is said to be*
a) proportional, if $U_i(D_i) \geq 1/n$ for each $i \in N$
b) exact, if $U_i(D_j) = 1/n$ for each $i, j \in N$
c) envy-free, if $U_i(D_i) \geq U_i(D_j)$ for each $i, j \in N$
d) equitable, if $U_i(D_i) = U_j(D_j)$ for each $i, j \in N$.

In many cases, the existence of a division with some property has only been proved at the cost of assigning to a player several nonadjacent intervals (like the result of Alon [1]). However, such a division may be very impractical in the real life. Therefore we are interested in cake divisions where each player receives a contiguous piece. Such cake divisions will be called *simple* and they are specified by their cutpoints and the order of players.

Definition 2 *A simple cake division is a pair $D = (d, \varphi)$, where d is an $(n-1)$-tuple $(x_1, x_2, ..., x_{n-1})$ of cutpoints with $0 \leq x_1 \leq x_2 \leq ... \leq x_{n-1} \leq 1$, and $\varphi : N \to N$ is a permutation of N.*

For technical reasons, we set $x_0 = 0$ and $x_n = 1$. Permutation φ in the division D means that player i is assigned the interval $[x_{j-1}, x_j]$ if $\varphi(j) = i$. Further, we shall suppose that each player i is indifferent between getting a piece with value zero and getting no piece in a division D; the latter occurs for a player i with $\varphi(j) = i$ and $x_{j-1} = x_j$.

The cutpoints $e_1, \dots, e_{n-1}$ of an equitable simple division (ESD for short) for n players and identity permutation fulfill $0 \leq e_1 \leq \cdots \leq e_{n-1} \leq 1$ and the following system of equalities:

$$
\begin{aligned}
F_1(e_1) &= F_2(e_2) - F_2(e_1) \qquad (1)\\
&= F_3(e_3) - F_3(e_2)\\
&\dots\\
&= F_{n-1}(e_{n-1}) - F_{n-1}(e_{n-2})\\
&= 1 - F_n(e_{n-1}).
\end{aligned}
$$

The main result of this paper is a proof showing that the equations system (1) always has a solution if the functions $F_i, i = 1, 2, \dots, n$ are nonnegative, nondecreasing and continuous.

Example 1. This example was presented by Hill and Morison in [7] as a counterexample for the claim of Brams, Jones and Klamler in [3].

The problems were caused by utility functions allowing zero intervals. However, we shall show that Hill and Morison were also wrong.
Let the probability densities of the three players be:

$$f_1(t) \equiv 1 \quad f_2(t) = \begin{cases} 3 & \text{if } t \in [0,1/3] \\ 0 & \text{if } t \in (1/3,1] \end{cases} \qquad f_3(t) = \begin{cases} 0 & \text{if } t \in [0,2/3] \\ 3 & \text{if } t \in (2/3,1] \end{cases}$$

For players' orders $(1,2,3),(2,1,3)$ and $(2,3,1)$ there are equitable cake divisions with (e_1,e_2) at positions $(1/4,11/12)$, $(1/5,4/5)$ and $(1/12,3/4)$, respectively, and the assigned values are equal to $1/4$, $3/5$ and $1/4$, respectively.

Further, if player 3 is first, then the only possibility for an ESD is that all players receive an equivalent of 0. Namely, if $e_1 \leq 2/3$ then player 3 necessarily gets a piece with value 0. On the other hand, if $e_1 > 2/3$, then player 2 cannot get a piece with positive value. So the only possibility for an ESD is to give each player 0. If players' order is (3,1,2) then this can be achieved by setting $1/3 \leq e_1 = e_2 \leq 2/3$; for (3,2,1) it is sufficient to set $e_2 = 1$ and to choose e_1 in the interval $[1/3,2/3]$ arbitrarily.

Finally, for the players' order $(1,3,2)$ an ESD can be achieved by setting $e_1 = 0$ and $e_2 \in [1/3,2/3]$. Player 1 receives no piece and the two other players pieces which they value at 0.

3 The main result

For two players, we are seeking a point $e_1 \in [0,1]$ such that

$$U_1(0,e_1) = U_2(e_1,1).$$

As the functions $F(x) = U_1(0,x)$ and $G(x) = U_2(x,1)$ of variable x on [0,1] are continuous and inequalities $F(0) \leq G(0), F(1) \geq G(1)$ hold, a simple application of the Intermediate Value Theorem implies the existence of a point e_1 with the desired properties.

However, as soon as we move to three players, difficulties arises. Suppose first, that player 3 has received the piece $[e_2,1]$. Players 1 and 2 want to divide the interval $[0,e_2]$ in an equitable manner. An equitable division point e_1 for two players on the interval $[0,e_2]$ always exists and it depends in a nondecreasing manner of e_2. So it seems that it suffices to find e_2 such that the common value U that players 1 and 2 incur on the interval $[0,e_2]$, as a function of variable e_2, is equal to the value assigned to player 3. The crucial step in the proof of the existence of such e_2 is to show that U is a nondecreasing and continuous function of e_2, which enables the use of Intermediate Value Theorem again.

This idea is hidden behind our proof. Formally, we use a notion of a generalized inverse (see citeEH10 or [5] to see how this defintion appears in the probability theory in the form of the quantile function) to capture the dependence of e_1 on e_2. We were able to prove immediately the case for an arbitrary number of players by mathematical induction.

Definition 3 *Let* $h : [0,1] \to [0,2]$ *be a nondecreasing function with* $h(1) \geq 1$. *The* generalized inverse *of* h *is a function* $h^- : [0,1] \to [0,1]$ *defined by*

$$h^-(x) := \inf\{z \in [0,1] : h(z) \geq x\}. \tag{2}$$

The properties of the generalized inverse are summarized in the following Lemma.

Lemma 1 *If* $h : [0,1] \to [0,2]$ *is a nondecreasing continuous function such that* $h(0) = 0$ *and* $h(1) \geq 1$ *then*

(i) $h^-(0) = 0$, $h^-(1) \leq 1$;

(ii) h^- *is nondecreasing;*

(iii) *for each* $x \in [0,1]$:

$$h^-(x) = \min\{z \in [0,1] : h(z) = x\} \quad \textit{hence} \quad h(h^-(x)) = x.$$

(iv) *For each* $x \in (0,1]$ *the left-hand limit*

$$h^-(x-) := \lim_{t \to x-} h^-(t)$$

exists, moreover,

$$h^-(x-) \leq h(x) \tag{3}$$

and

$$h(h^-(x-)) = x. \tag{4}$$

Further, for each $x \in [0,1)$ *the right-hand limit*

$$h^-(x+) := \lim_{t \to x+} h^-(t)$$

exists, moreover,

$$h^-(x) \leq h^-(x+) \tag{5}$$

and

$$h(h^-(x+)) = x.$$

(v) *Function* h^- *is continuous from the left in each* $x \in (0,1]$.

Proof.
(i) Trivial.
(ii) It suffices to realize that

$$x_1 < x_2 \text{ implies } \{z \in [0,1] : h(z) \geq x_2\} \subseteq \{z \in [0,1] : h(z) \geq x_1\}.$$

(iii) Let us define for each $x \in [0,1]$:

$$M_x = \{z \in [0,1] : h(z) = x\}, \qquad N_x = \{z \in [0,1] : h(z) \geq x\}.$$

Let $x \in [0,1]$. The set M_x is a nonempty closed interval, say $M_x = [a,b]$. Similarly, N_x is a nonempty closed interval, say $N_x = [c,d]$. As $M_x \subseteq N_x$, we have $c \leq a$. Further, $x \leq h(c) \leq h(a) = x$, since h is nondecreasing. This implies $c \in M_x$, hence $c \geq a$. The rest follows.
(iv) The existence of one-sided limits and inequalities (3) and (5) follow from (ii). To show (4), let $x \in (0,1]$ be fixed and take any sequence $x_n \in (0,x)$ such that

$$\lim_{n\to\infty} x_n = x.$$

Then

$$\lim_{n\to\infty} h^-(x_n) = h^-(x-).$$

Property (iii) implies that $h(h^-(x_n)) = x_n$ for each $n \in \mathbb{N}$. Using continuity of h, taking this equality to the limit (for $n \to \infty$) gives

$$h(h^-(x-)) = x.$$

Equality $h(h^-(x+)) = x$ can be shown similarly.
(v) Equality $h(h^-(x-)) = x$ implies $h^-(x-) \in M_x$, hence $h^-(x) \leq h^-(x-)$ and the left continuity follows. ■

Lemma 2 *Let $g, f : [0,1] \to [0,1]$ be continuous nondecreasing functions such that $g(0) = f(0) = 0$, $f(1) = 1$. Then*

(i) *the function $g \circ (g+f)^-$ is continuous.*

(ii) *for each $y \in [0,1]$ there exists $x \in [0,y]$ such that $g(x) + f(x) = f(y)$.*

Proof. (i) Let us denote the function $g + f$ by h and take any $x \in [0,1)$. Since both f, g are nondecreasing, equalities

$$\begin{aligned} g(h^-(x)) + f(h^-(x)) &= h(h^-(x)) = x, \\ g(h^-(x+)) + f(h^-(x+)) &= h(h^-(x+)) = x, \end{aligned}$$

imply

$$g(h^-(x)) = g(h^-(x+)).$$

Function g is continuous, so $(g \circ h^-)(x+) = g(h^-(x+)) = g(h^-(x))$ and the continuity of $g \circ h^-$ from the right follows.

Continuity from the left is implied by Lemma 1 (v).
(ii) Let $y \in [0,1]$. Function $g + f$ is continuous on $[0,y]$, and

$$\begin{aligned} g(0) + f(0) &\leq f(y), \\ g(y) + f(y) &\geq f(y). \end{aligned}$$

The Intermediate Value Theorem now implies that $x \in [0,y]$ exists such that $g(x) + f(x) = f(y)$. ■

Let us now return to the equation system (1). Define the functions $G_j : [0,1] \to [0,1]$ recursively by

$$G_1 := F_1, \qquad G_j := G_{j-1} \circ (G_{j-1} + F_j)^- \circ F_j \text{ for } j = 2, \ldots, n-1.$$

The following assertion can be proved by induction.

Lemma 3 *If for each $k = 1, 2, \ldots, n$ functions $F_k : [0,1] \to [0,1]$ are nondecreasing, continuous and surjective, then all the functions G_k exist, are nondecreasing and continuous, moreover, $G_k(0) = 0$.*

Set $e_n := 1$. Then $F_n(e_n) = 1$. Lemma 2 (ii) implies that there exist

$$\begin{array}{rcl}
e_{n-1} & \in & [0, e_n] \text{ such that } \; G_{n-1}(e_{n-1}) + F_n(e_{n-1}) = F_n(e_n); \\
e_{n-2} & \in & [0, e_{n-1}] \text{ such that } \; G_{n-2}(e_{n-2}) + F_{n-1}(e_{n-2}) = F_{n-1}(e_{n-1}); \\
\ldots & & \\
e_{j-1} & \in & [0, e_j] \text{ such that } \; G_{j-1}(e_{j-1}) + F_j(e_{j-1}) = F_j(e_j); \\
\ldots & & \\
e_1 & \in & [0, e_2] \text{ such that } \; G_1(e_1) + F_2(e_1) = F_2(e_2);
\end{array}$$

Moreover, we claim that for each $j = 2, 3, \ldots, n-1$:

$$G_j(e_j) = G_{j-1}(e_{j-1})$$

Let us denote

$$x_{j-1} = ((G_{j-1} + F_j)^- \circ F_j)(e_j). \tag{6}$$

Then

$$\begin{aligned}
G_{j-1}(x_{j-1}) + F_j(x_{j-1}) &= (G_{j-1} + F_j)(x_{j-1}) \\
&= ((G_{j-1} + F_j) \circ (G_{j-1} + F_j)^- \circ F_j)(e_j) = F_j(e_j)
\end{aligned}$$

by Lemma 1(iii). So we have

$$G_{j-1}(x_{j-1}) + F_j(x_{j-1}) = F_j(e_j),$$

which together with the equality

$$G_{j-1}(e_{j-1}) + F_j(e_{j-1}) = F_j(e_j)$$

implies $G_{j-1}(x_{j-1}) = G_{j-1}(e_{j-1})$, as functions G_{j-1} and F_j are nondecreasing. After substituting (6) for x_{j-1} and using the definition of function G_j, this implies

$$G_{j-1}(e_{j-1}) = G_{j-1}(x_{j-1}) = G_{j-1}((G_{j-1} + F_j)^- \circ F_j)(e_j) = G_j(e_j)$$

and so the claim is proved.
Using the claim and equality $F_1 = G_1$, we get that for each $j = 1, 2, \ldots, n$:

$$F_1(e_1) = F_j(e_j) - F_j(e_{j-1})$$

and we have proved that a solution $0 \leq e_1 \leq e_2 \leq \cdots \leq e_{n-1} \leq 1$ for the system of equations (1) exists.

Translating this result into the language of cake cutting and taking into account that the same argument can be made for any other player's order, we get as our main result the following assertion.

Theorem 1 *For any number of players n there exists a simple equitable division with each players' order.*

We can also consider the case when the probability density functions of all players are strictly positive. Then one can add to the assumptions of Lemma 2 that function f is strictly increasing and strengthen assertion (ii) to saying that for each $y \in [0, 1]$ a *unique* point x fulfilling the desired equality exists. Applying this stronger assertion, we get that the inductively defined points $e_{n-1}, e_{n-2}, \ldots, e_1$ are also uniquelly determined, so the previous theorem can also be formulated in a stronger way.

Theorem 2 *Let the probability density function of each among n players be strictly positive. Then for any players' order a unique simple equitable division exists.*

4 Equitability, proportionality and envy-freeness

In the previous section we have proved that an equitable simple division exists for any players' order. However, in an equitable division all players may get as little as zero, as demonstrated by Example 1. Such a division might be fair, but for sure not very efficient. Fortunately, as shown in [9], there always exists a players' order π, in which a proportional simple division exists. Arranging players according to π, a simple division exists that is simultaneously proportional and equitable.

Lemma 4 *Let D' be any simple cake division for n players with players' order π such that player i receives a piece with value $U_i(D'_i)$. Then any ESD D with players' order π fulfills*

$$U_i(D_i) \geq min\{U_j(D'_j), j = 1, 2, ..., n\}.$$

Proof. To avoid an awkward work with indices, let us suppose, without loss of generality, that π is the identity permutation. Denote the cutpoints of the division D' by $(d_1, d_2, ..., d_{n-1})$ and the cutpoints of the equitable cake division D in the players' order π by $(e_1, e_2, \ldots, e_{n-1})$. Since D is equitable, it suffices to show that $U_i(D_i) \geq U_i(D'_i)$ for some i.

We distinguish three cases:

(a) $e_1 \geq d_1$. Then $[0, d_1] \subset [0, e_1]$ and so $U_1(D_1) \geq U_1(D'_1)$.

(b) If there exists k such that $e_j < d_j$ for each $j = 1, 2, ..., k-1$ and $e_k \geq d_k$ then $[d_{k-1}, d_k] \subset [e_{k-1}, e_k]$. Hence $U_k(D_k) \geq U_k(D'_k)$.

(c) $e_j < d_j$ for each $j = 1, 2, ..n-1$. Then $[d_{n-1}, 1] \subset [e_{n-1}, 1]$ therefore $U_n(D_n) \geq U_n(D'_n)$.

■

Theorem 3 *For any number of players n there exists a players' order that admits a simple division that is simultaneously proportional and equitable.*

Example 2. On the other hand, there exist players' valuations that do not admit an ESD that is simultaneously envy free. This example is taken from Brams, Jones and Klamler [3]. To be self-contained, we give here a complete argument to show the claim.
The probability density of player 1 is everywhere equal to 1, for players 2 and 3 we have:

$$f_2(t) = \begin{cases} -4t + 2 & \text{if } t \in [0, 1/2] \\ 4t + 2 & \text{if } t \in (1/2, 1] \end{cases} \qquad f_3(t) = \begin{cases} -2t + 3/2 & \text{if } t \in [0, 1/2] \\ 2t - 1/2 & \text{if } t \in (1/2, 1] \end{cases}$$

Let us suppose that the the cutpoints are x and y. As the densities are symmetric, we can simplify the notation by considering interval $[0, y]$ instead of the interval $[y, 1]$ that was assigned to the right player. Further, since the left player is not allowed to envy the right player and conversely, we must have $y = x$.

Suppose that player 1 is in the middle. Equitability for players 2 and 3 implies

$$\int_0^x (-4t + 2)dt = -2x^2 + 2x = \int_0^x (-2t + 3/2)dt = -x^2 + 3x/2.$$

This is only possible if $x = 0$ (in this case player 1 would receive the whole interval) or if $x = 1/2$ (now player 1 gets 0). So in neither case player 1 can achieve the same value as the others two.

Hence the middle piece must be given either to player 2 or to player 3. The former case leads to the equation

$$\int_0^x (-2t + 3/2)dt = -x^2 + 3x/2 = x$$

and the latter to

$$\int_0^x (-4t+2)dt = -2x^2 + 2x = x.$$

These equations have the roots $x = 0, 1/2$ and $x = 0, 1$, respectively, which again does not correspond to an equitable division.

5 Conclusion

In this paper we have proved that for any number of players an equitable simple division (ESD) exists for any players' order. This implies that for at least one players' order an ESD exists that simultaneously fulfills proportionality. However, we used an example of [3] to recall that in some cases no ESD exists that is also envy-free.

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