Mathematical Spectrum

A magazine for students and teachers of mathematics
in schools, colleges and universities,
and for everyone interested in mathematics

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Mathematical Spectrum is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

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The Editorial Committee welcomes the submission of suitable material, including correspondence, queries and solutions to problems, for publication in *Mathematical Spectrum*. Students are encouraged to send in contributions; prizes are available annually for students as follows: up to the value of £50 for articles, and up to £50 for letters, solutions to problems and other items. All correspondence should be sent to

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Some Irrationals do not have Special Names

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The proof that $\sqrt{2}$ is irrational appears in textbooks as a standard introduction to proof by contradiction. Numbers like $\sqrt{2}$ fall into a special category of irrational numbers of the form $\sqrt[n]{a}$, where a is not a perfect nth power. These numbers can be seen as the solution to $x^n = a$. Other irrational numbers can be easily obtained as solutions to polynomial equations. For example, $\sqrt{2}+\sqrt{3}$ and $2\cos(\pi/7)$ are solutions to $x^4-10x^2+1=0$ and $x^3+x^2-2x-1=0$ respectively. The irrationality of these numbers follows from the rational root theorem (see reference 1). Yet more irrational numbers arise when you solve nonalgebraic equations. Usually they do not have special names. The aim of this article is to present an elementary approach to irrationality of roots of some charming equations of this type.

Example 1 (See reference 2.) Evidently, $x = 3$ is a solution to

$$3^x = x^3. \tag{1}$$

But there is another real number that satisfies this equation and is irrational.

It existence follows from Bolzano's theorem, which says that a continuous function cannot change its sign without going through the zero. (This result is also known as the mean value theorem.) Put

$$f(x) = 3^x - x^3.$$

Since $f(2) > 0$ while $f(\frac{5}{2}) < 0$, and f is a continuous function, there must be some number x between 2 and $\frac{5}{2}$ with the property $f(x) = 0$.

Since for each real number x we have $3^x > 0$, (1) is equivalent to

$$\frac{\ln x}{x} = \frac{\ln 3}{3}.$$

Since

$$\left(\frac{\ln x}{x}\right)' = \frac{1-\ln x}{x^2},$$

the function $\ln x/x$ is increasing on the interval $(0, \mathrm{e})$ and it is decreasing for $x > \mathrm{e}$.

Therefore, (1) has exactly one root in the interval $(0, \mathrm{e})$, denoted by z, say, and it has exactly one root larger than e, namely $x = 3$, as we know.

Now we show that z is irrational by contradiction. Assume that

$$z = \frac{p}{q},$$

where p and q are positive integers and p/q is in lowest terms. Substituting p/q for x into (1) gives

$$3^{p/q} = \left(\frac{p}{q}\right)^3,$$
$$3^p = \left(\frac{p}{q}\right)^{3q},$$
$$3^p q^{3q} = p^{3q}. \tag{2}$$

Hence, p^{3q} is divisible by 3. This means that p must be as well. Substituting $p = 3r$, where r is a positive integer, into (2) gives

$$3^{3r} q^{3q} = (3r)^{3q}$$
$$q^{3q} = 3^{3(q-r)} r^{3q}. \tag{3}$$

Since $z < 3$, we have $r < q$, which yields that the right-hand side of (3) is divisible by 3. So q^{3q} is divisible by 3. This means that q must be as well. But this also means that p and q are both divisible by 3, so p/q was not in lowest terms, which is a contradiction. The numerical value of z is 2.478 052 680 288 . . . .

Example 2 (See references 3 and 4.) It is clear that $x = \frac{1}{2}$ and $x = \frac{1}{4}$ are solutions to

$$a^{a^x} = x, \tag{4}$$

where $a = \frac{1}{16}$. But there is another real number that satisfies this equation and is irrational.

Since $a^x = x$ implies $a^{a^x} = x$, we start with the equation

$$a^x = x. \tag{5}$$

Put

$$f(x) = a^x - x.$$

Since $f(\frac{1}{4}) = \frac{1}{4} > 0$ while $f(\frac{1}{2}) = -\frac{1}{4} < 0$, and f is a continuous function, there must be some number x between $\frac{1}{4}$ and $\frac{1}{2}$ with the property $f(x) = 0$. The reason for this number being unique is that the function f is decreasing (as a sum of two decreasing functions). Therefore, (5) has exactly one root, denoted by z, say.

Now we show by contradiction that z is irrational. Assume that

$$z = \frac{p}{q},$$

where p and q are positive integers and p/q is in lowest terms. Substituting p/q for x into (5) gives

$$a^{p/q} = \frac{p}{q},$$

or

$$q^q = 2^{4p} p^q. \tag{6}$$

So q^q is even. This means that q must be even as well. Substituting $q = 2r$, where r is a positive integer, into (6) gives

$$\begin{aligned}(2r)^{2r} &= 2^{4p} p^{2r},\\ r^{2r} &= 2^{4p-2r} p^{2r}.\end{aligned} \tag{7}$$

Since $z > \frac{1}{4}$, we have $4p > 2r$, which yields that the right-hand side of (7) is even. So r^{2r} is even. This means that r must be even as well.

Substituting $r = 2s$, where s is a positive integer, into (7) gives

$$\begin{aligned}(2s)^{4s} &= 2^{4p-4s} p^{4s},\\ 2^{8s-4p} s^{4s} &= p^{4s}.\end{aligned} \tag{8}$$

Since $z < \frac{1}{2}$, we have $4p < 8s$, which yields that the left-hand side of (8) is even. So p^{4s} is even. This means that p must be even as well.

So p and q are both even which is a contradiction to our assumption that they have no common factors. The numerical value of z is $0.364\,249\,889\,784\ldots$.

Finally, we show that (4) has no other roots. Since for each real number x we have $a^x > 0$, (4) is equivalent to

$$f(x) = 0, \tag{9}$$

where

$$f(x) = 16^x \ln x + \ln 16.$$

Then

$$f'(x) = \frac{16^x}{x} g(x),$$

where

$$g(x) = x \ln x \ln 16 + 1.$$

It is not difficult to verify that the function $g(x)$ is decreasing on $(0, 1/\mathrm{e})$ and increasing for $x > 1/\mathrm{e}$. Since $g(\frac{1}{4}) = g(\frac{1}{2}) > 0$ and $g(1/e) < 0$, there are positive real numbers a and b such that $g(x) < 0$ for each $x \in (a, b)$ and $g(x) > 0$ for each $x \in (0, a) \cup (b, \infty)$. Therefore, the function $f(x)$ is decreasing on the interval $[a, b]$ and it is increasing on the intervals $(0, a]$ and $[b, \infty)$. This yields that (9) can have at most three roots.

References

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