Out of l'Hôpital clutches: an example

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How to establish the limit $\lim_{x\to 0}(x - \sin x)/(x^3)$ before learning about l'Hôpital rule? The aim of this note is to answer this very reasonable question.

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Some 30 years ago, the author first met the following limit

$$\lim_{x\to 0}\frac{x - \sin x}{x^3}.$$

It was the first time he was unable to solve a limit without using derivatives. But with l'Hôpital rule it was too easy!

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Professor Cavallini [1] shows that it follows immediately from the following property

$$9f(3x) - f(x) = \frac{4}{3}\left(\frac{\sin x}{x}\right)^3$$

of the function

$$f(x) = \frac{x - \sin x}{x^3},$$

that $\lim_{x\to 0} f(x)$, if it exists, is necessarily $1/6$.

The author found independently a similar result (unpublished) based on the identity

$$4f(2x) - f(x) = \frac{1}{1+\cos x}\left(\frac{\sin x}{x}\right)^3.$$

However, we still cannot conclude anything as the limit may or may not exist. Indeed, if we apply previous method to the function

$$f(x) = \cos\frac{1}{x},$$

it follows from the properties

$$f\left(\frac{x}{2}\right) = 2f^2(x) - 1, \quad f\left(\frac{x}{3}\right) = 4f^3(x) - 3f(x),$$

that $\lim_{x\to 0} f(x)$, if it exists, is necessarily 1. But it is well known that this limit does not exist.

The existence of the limit $\lim_{x\to 0}(x - \sin x)/(x^3)$ is not easy to show. We start from the following identity, which is proved in [2]

$$\sum_{n=1}^{\infty} 3^{n-1}\sin^3\left(\frac{x}{3^n}\right) = \frac{x - \sin x}{4},$$

which yields (for $x \neq 0$)

$$\sum_{n=1}^{\infty} f_n(x) = \frac{x - \sin x}{x^3}, \tag{1}$$

where

$$f_n(x) = \frac{4}{x^3}\cdot 3^{n-1}\sin^3\left(\frac{x}{3^n}\right).$$

Using the basic limit $\lim_{\theta\to 0}(\sin\theta)/\theta = 1$ [3], we reconstruct the function f_n so that

$$f_n(0) = \lim_{x\to 0} f_n(x) = \lim_{x\to 0}\frac{4}{3^{2n+1}}\cdot\frac{\sin^3(x/3^n)}{(x/3^n)^3} = \frac{4}{3^{2n+1}}$$

and thus remove its discontinuity.

It is not difficult to verify that

$$\sum_{n=1}^{\infty} f_n(0) = \sum_{n=1}^{\infty} \frac{4}{3^{2n+1}} = \frac{1}{6}. \tag{2}$$

Since $|\sin\theta| \leqq |\theta|$, we have

$$|f_n(x)| \leqq \frac{4}{3^{2n+1}}. \tag{3}$$

Therefore, by the Weierestrass M-test, $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly.

Since the sum of this series is a continous function, by (1) and (2) we obtain

$$\lim_{x\to 0} \frac{x - \sin x}{x^3} = \frac{1}{6}. \tag{4}$$

Now we are ready to give an epsilon–delta proof of (4).

Let $\varepsilon > 0$. Choose $n \in \mathbb{N}$ such that $1/3^{2n+1} < \varepsilon$. Put

$$S_k(x) = \sum_{i=1}^{k} f_i(x). \tag{5}$$

Since $4\sin^3\theta = 3\sin\theta - \sin 3\theta$, by using the squeeze principle we obtain

$$S_k(x) = \frac{1}{x^3}\sum_{i=1}^{k}\left[3^i \sin\left(\frac{x}{3^i}\right) - 3^{i-1}\sin\left(\frac{x}{3^{i-1}}\right)\right] = \frac{1}{x^3}\cdot\left(3^k \sin\frac{x}{3^k} - \sin x\right), \tag{6}$$

which yields

$$\lim_{k\to\infty} S_k(x) = \frac{x - \sin x}{x^3}. \tag{7}$$

Indeed, by using the basic limit $\lim_{\theta\to 0} (\sin\theta)/\theta = 1$, we obtain $\lim_{k\to\infty} 3^k \sin(x/3^k) = x$.

By (3) we have

$$\left|S_{n+k}(x) - S_n(x)\right| \leqq \sum_{i=n+1}^{n+k} \left|f_i(x)\right| \leqq \sum_{i=n+1}^{n+k} \frac{4}{3^{2i+1}} = \frac{1}{2}\cdot\frac{1}{3^{2n+1}}\cdot\left(1 - \frac{1}{3^{2k}}\right).$$

Therefore

$$\lim_{k\to\infty} |S_{n+k}(x) - S_n(x)| \leqq \lim_{k\to\infty} \frac{1}{2}\cdot\frac{1}{3^{2n+1}}\cdot\left(1 - \frac{1}{3^{2k}}\right),$$

and by (7) we have

$$\left|\frac{x - \sin x}{x^3} - S_n(x)\right| \leqq \frac{1}{2}\cdot\frac{1}{3^{2n+1}}.$$

Since

$$\lim_{x\to 0} S_n(x) = \lim_{x\to 0} \sum_{i=1}^{n} f_i(x) = \sum_{i=1}^{n} \frac{4}{3^{2i+1}} = \frac{1}{6} - \frac{1}{2}\cdot\frac{1}{3^{2n+1}},$$

for $\varepsilon_1 = \varepsilon - 1/3^{2n+1} > 0$ there is $\delta > 0$ such that for each $x \in (-\delta, \delta)$, $x \neq 0$, we have

$$\left| S_n(x) - \left(\frac{1}{6} - \frac{1}{2} \cdot \frac{1}{3^{2n+1}} \right) \right| < \varepsilon_1 = \varepsilon - \frac{1}{3^{2n+1}}.$$

Thus

$$\left| \frac{x - \sin x}{x^3} - \frac{1}{6} \right| \leqq \left| \frac{x - \sin x}{x^3} - S_n(x) \right| + \left| S_n(x) - \left(\frac{1}{6} - \frac{1}{2} \cdot \frac{1}{3^{2n+1}} \right) \right| + \frac{1}{2} \cdot \frac{1}{3^{2n+1}}$$
$$< \frac{1}{2} \cdot \frac{1}{3^{2n+1}} + \left(\varepsilon - \frac{1}{3^{2n+1}} \right) + \frac{1}{2} \cdot \frac{1}{3^{2n+1}} = \varepsilon.$$

The proof is complete.

References

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