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SUMS OF CLOSED GRAPH FUNCTIONS

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Dedicated to Professor T. Šalát on the occasion of his 70th birthday

ABSTRACT. The main purpose of this paper is to show that each Baire one star function $f\colon \mathbb{R} \to \mathbb{R}$ can be expressed as the sum of two closed graph functions.

We say that a function $f\colon \mathbb{R} \to \mathbb{R}$ *has a closed graph* (written $f \in \mathcal{U}$) if the graph of the function f, i.e., the set $\{(x,y) \in \mathbb{R} \times \mathbb{R}\colon y = f(x)\}$ is a closed subset of the product $\mathbb{R} \times \mathbb{R}$.

PROPOSITION A. (See [1].) *Let $f, g \in \mathcal{U}$ be nonnegative functions. Then $f + g \in \mathcal{U}$.*

The following example shows that the assumption "nonnegative" in Proposition A cannot be omitted.

EXAMPLE. (See [5].) Define $f, g\colon \mathbb{R} \to \mathbb{R}$ as follows

$$f(x) = \begin{cases} 1, \\ \frac{1}{x}, \end{cases} \qquad g(x) = \begin{cases} 0 & \text{if } x = 0, \\ -\frac{1}{x} & \text{otherwise}. \end{cases}$$

Evidently $f, g \in \mathcal{U}$ and $f + g \notin \mathcal{U}$.

A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be *a Baire-one-star function* (written $f \in \mathcal{B}_1^*$) if for any nonempty closed $F \subset \mathbb{R}$ there is a nonvoid $G \subset F$, relatively open in F, such that the restriction of f to F is continuous on G.

We say that a function $f\colon \mathbb{R} \to \mathbb{R}$ is *piecewise continuous* (see [3]), if there exist closed subsets $F_1, F_2, \dots$ of $\mathbb{R}$ such that $\bigcup_{i=1}^{\infty} F_i = \mathbb{R}$ and each of the restrictions $f|F_i$ is continuous. It is known that a function $f\colon \mathbb{R} \to \mathbb{R}$ is piecewise continuous iff it is a Baire-one-star function (see [4]).

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PROPOSITION B. (See [2].) *Let a function* $f\colon \mathbb{R} \to \mathbb{R}$ *have a closed graph. If* K *is a compact subset of* $\mathbb{R}$ *then* $f^{-1}(K)$ *is a closed subset of* $\mathbb{R}$.

It is not difficult to verify that each closed graph function $f\colon \mathbb{R} \to \mathbb{R}$ is piecewise continuous (where we put $F_i = \{x \in \mathbb{R}\colon |f(x)| \le i\}$).

THEOREM 1. *For each* $f \in \mathcal{B}_1^*$ *there exist nonnegative* $f_1, f_2 \in \mathcal{U}$ *such that*

$$f = f_1 - f_2 .$$

Proof. Let $\{F_i\}_{i=1}^{\infty}$ be an increasing sequence of closed subsets of $\mathbb{R}$ such that $\bigcup_{i=1}^{\infty} F_i = \mathbb{R}$ and the restrictions of f to the sets $F_1, F_2, \dots$ are continuous. Put $E_0 = F_1$, $E_i = F_{i+1} - F_i$ for each $i \in \mathbb{N}$. Define $g\colon \mathbb{R} \to \mathbb{R}$ as follows

$$g(x) = \begin{cases} \frac{1}{\operatorname{dist}(x,F_i)} & \text{if } x \in E_i\,,\ i \in \mathbb{N}, \\ 0 & \text{if } x \in E_0\,. \end{cases}$$

We show that $g \in \mathcal{U}$. Let $x_0 \in \mathbb{R}$, $x_n \to x_0$, $g(x_n) \to y_0$. Let $k \in \mathbb{N}$ be such that $x_0 \in E_k$. Then there exists $n_0 \in \mathbb{N}$ such that $x_n \in F_{k+1}$ for each $n \ge n_0$. Indeed, if $x \in \{x_n\colon n \in \mathbb{N}\}$ be such that $x \notin F_{k+1}$, then $x \in E_i$ for some $i \ge k+1$, which yields

$$g(x) = \frac{1}{\operatorname{dist}(x, F_i)} \ge \frac{1}{|x - x_0|}\,.$$

Since $x_n \to x_0$ and $g(x_n) \to y_0$, the set of all such x must be finite.

In the case $k = 0$ evidently $g(x_n) = 0 = g(x_0)$, thus $g(x_n) \to g(x_0)$. Suppose that $k > 0$. Then $x_0 \in F_{k+1} - F_k$. Since F_k is closed, there exists $n_1 \in \mathbb{N}$ such that $x_n \notin F_k$ for each $n \ge n_1$. Therefore $x_n \in E_k$ for each $n \ge n_0 + n_1$. Since $g|E_k$ is continuous, we have $g(x_n) \to g(x_0)$.

Put $f_1 = g + \max(f, 0)$, $f_2 = f_1 - f$. It is not difficult to verify that $f_1, f_2 \in \mathcal{U}$ (in the same way as for the function g). □

Since $\mathcal{B}_1^*$ is closed with respect to addition, we have the following

COROLLARY. $\mathcal{U} + \mathcal{U} = \mathcal{B}_1^*$.

In the next part we will study discontinuity points of such functions. Notice that $f \in \mathcal{B}_1^*$ iff for every nonempty closed $F \subset \mathbb{R}$ the set $D(f|F)$ (i.e., the set of discontinuity points of the function $f|F$) is nowhere dense in F.

Let $f \in \mathcal{B}_1^*$. Put $E_0 = F_0 = \mathbb{R}$. For each ordinal number $\xi \ge 1$ put

$$E_\xi = \bigcap_{\eta < \xi} F_\eta\,, \qquad F_\xi = \operatorname{Cl} D(f|E_\xi)\,.$$

Then there is the smallest (countable) ordinal number $\xi(f)$ such that $F_{\xi(f)} = \emptyset$.

On the other hand, for each countable ordinal number ξ there exists a function $f \in \mathcal{U}$ such that $\xi(f) = \xi$. Indeed, using a binary Cantor tree (see [6]) it is not difficult to verify that there is a decreasing sequence $\{F_\eta\}_{\eta\le\xi}$ of nonempty closed subsets of the standard Cantor set such that the set F_β is nowhere dense in F_α for each $\alpha < \beta \le \xi$. Put $E_0 = \mathbb{R}$, $E_\xi = \bigcap_{\eta<\xi} F_\eta$ for $\eta \ge 1$. Define $f\colon \mathbb{R} \to \mathbb{R}$ as follows

$$f(x) = \begin{cases} \frac{1}{\operatorname{dist}(x,F_\eta)} & \text{if } x \in E_\eta - F_\eta \ (\eta \le \xi), \\ 0 & \text{otherwise.} \end{cases}$$

THEOREM 2. *For each $f \in \mathcal{B}_1^*$ there exist nonnegative $f_1, f_2 \in \mathcal{U}$ such that*

$$f = f_1 - f_2 \quad \text{and} \quad D(f_1) \cup D(f_2) \subset \operatorname{Cl} D(f).$$

Proof. It is sufficient to define $g\colon \mathbb{R} \to \mathbb{R}$ as follows

$$g(x) = \begin{cases} \frac{1}{\operatorname{dist}(x,F_\eta)} & \text{if } x \in E_\eta - F_\eta\big(\eta \le \xi(f)\big), \\ 0 & \text{otherwise.} \end{cases}$$

□

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