TECHNICKÁ UNIVERZITA V KOŠICIACH

STROJNÍCKA FAKULTA

1952 - 1997

ZBORNÍK VEDECKÝCH PRÁC

z medzinárodnej konferencie, poriadanej pri príležitostí
45. výročia založenia
Strojníckej fakulty TU v Košiciach

NOVÉ TRENDY V STROJÁRSTVE NA PRAHU TRETIEHO TISÍCROČIA

1. sekcia
MATEMATIKA A JEJ APLIKÁCIE V TECHNICKÝCH VEDÁCH

Herľany, 25. a 26.9.1997

ISBN 80-967636-7-9

Obsah

On almost continuous functions

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ABSTRACT. We study the well-known notions of Darboux property, almost continuity and closed graph function, all for functions with the domain $\mathbf{R}$ and the range a subset of $\mathbf{R}$. We prove for each closed graph function f that the function $h: h(x) = \sin(f(x))$ is almost continuous as a corollary to a general theorem characterizing all continuous functions g for which, being substituted for sin, h remains almost continuous.

1. Introduction

Throught all this paper, the notion "function" is used in the sense "function with the domain $\mathbf{R}$ and the range a subset of $\mathbf{R}$". We identify the function with its graph (as a subset of the plane).

This note is motivated before all by [2], Theorem 2 and Theorem 3. both regarding quasicontinuity. Our main goal here is to prove analogical results for (Darboux property and) almost continuity presented in the following Theorem and Corollary.

Theorem. *Let g be any continuous function. Then the conditions $(*)$ and $(**)$ are equivalent for g, where*

$(*)$ *for each open set V such that $g^{-1}[V] \neq \emptyset$,*

$$\sup g^{-1}[V] = \infty \text{ and } \inf g^{-1}[V] = -\infty,$$

$(**)$ *for each closed graph function f,*

the function $h: h(x) = g(f(x))$ is almost continuous.

Corollary (for $g = \sin$). *Let f be any closed graph function. Then the function $h: h(x) = \sin(f(x))$ is almost continuous.*

Again originally our result and the proof were given for the function $g: g(x) = \sin x$, but except of the continuity of g we need and use only the property $(*)$ of the function g. The key role in the proof of our Theorem play

- [1], Theorem 2, allowing to deal with the Darboux property instead of the almost continuity,
- [2], Theorem 1,
- [3], Theorem 9.

Before presenting them, let us recall some basic notions and facts (most of them well-known; (E) we adopt from [1]).

Definition. Let f be any function.

- f is almost continuous, if for every open set G (a subset of the plane) containing f there exists a continuous function g contained in G,
- f has the Darboux property, if for every interval I, $f[I]$ is an interval, too,
- f satisfies the condition (E), if for every $a \in \mathbb{R}$, $[a, f(a)]$ is a point of the boundary of $f \restriction (-\infty, a)$ and of the boundary of $f \restriction (a, \infty)$.

It is known (see e. g. [1]) that the implications

$$f \text{ is almost continuous } \Rightarrow f \text{ has a connected graph } \Rightarrow$$
$$\Rightarrow f \text{ has the Darboux property } \Rightarrow f \text{ satisfies the condition (E)}$$

hold for each function f, but no converse implication is true in general. Another basic fact (see e. g [4]) useful here is the following:

if g is any continuous and f any Baire class 1 function, then the function $h: h(x) = g(f(x))$ is in the Baire class 1, too.

Finally let us present the promising key statements leading to our results.

[1], **Theorem 2.** Let h be any Baire class 1 function. Then if h satisfies the condition (E), then h is almost continuous.

[2], **Theorem 1.** Let f be any closed graph function, let a be a point of discontinuity of f from the left (right). Then for each left (right) neighborhood U of a there exists an interval $J \subseteq U$ such that f is continuous and unbounded on J.

[3], **Theorem 9.** Let f be any closed graph function. Then f is in the Baire class 1.

2. Proof of the Theorem

a) Proof of $(*) \Rightarrow (**)$. Clearly it suffices to prove the

Lemma. *Let g be any continuous function satisfying $(*)$, let f be any closed graph function. Then the function $h: h(x) = g(f(x))$ has the Darboux property.*

The proof of the Lemma uses the following

Proposition. *Let g be any function with the Darboux property satisfying $(*)$, let f be any function which is continuous and unbounded on the interval J. Then* $\operatorname{int}(\operatorname{rng} g) \subseteq g[f[J]]$.

Proof of the Proposition. Take arbitrary $y \in \operatorname{int}(\operatorname{rng} g)$ and $\varepsilon > 0$ such that $(y - \varepsilon, y + \varepsilon) \subseteq \operatorname{rng} g$. Let us consider the case f unbounded above on J, the opposite case is analogical. Since f is continuous and unbounded above on J, there exists $a \in \mathbb{R}$ such that $(a, \infty) \subseteq f[J]$, hence $g[(a, \infty)] \subseteq g[f[J]]$. Now it suffices to show that $y \in g[(a, \infty)]$, i. e. that $(\exists\, x > a)(g(x) = y)$: since

$$g^{-1}[(y - \varepsilon, y)] \neq \emptyset \neq g^{-1}[(y, y + \varepsilon)],$$

we have

$$\sup g^{-1}[(y-\varepsilon, y)] = \infty = \sup g^{-1}[(y, y+\varepsilon)].$$

Therefore there exist $x_1, x_2 > a$ such that $g(x_1) < y$ & $g(x_2) > y$. The Darboux property of g gives the required $x > a$, which completes the proof of the Proposition.

Proof of the Lemma. Let I be an interval; we show that $h[I]$ is an interval, too. We distinguish two cases.
Case (i). $f \restriction I$ is discontinuous at some point of I. By [2], Theorem 1, there is an interval $J \subseteq I$ such that f is continuous and unbounded on J. By the Proposition we have

$$\operatorname{int}(\operatorname{rng} g) \subseteq g[f[J]] \subseteq g[f[I]] \subseteq \operatorname{rng} g,$$

i. e. $\operatorname{int}(\operatorname{rng} g) \subseteq h[I] \subseteq \operatorname{rng} g$. Since g is continuous, $\operatorname{rng} g$ is an interval, therefore so is $h[I]$.
Case (ii). $f \restriction I$ is continuous. Since g is continuous, $h \restriction I$ is a composition of two continuous functions, hence $h[I]$ is again an interval, which completes the proof of the Lemma.

b) Proof of $\neg(*) \Rightarrow \neg(**)$. Let V be an open set such that $g^{-1}[V] \neq \emptyset$, $s = \sup g^{-1}[V] < \infty$ and choose arbitrary fixed $b \in g^{-1}[V]$. (The case $\inf g^{-1}[V] > -\infty$ is analogical.) Take the closed graph function f as follows: put $f(0) = b$ and for $x \neq 0$ put $f(x) = s + \frac{1}{|x|}$. We have $g(f(0)) = g(b) \in V$ (V open!), but for $x \neq 0$ $f(x) > s$, thus $g(f(x)) \notin V$. Therefore h clearly fails (E) at $a = 0$, because $[0, h(0)] \in \mathbf{R} \times V$ ($\mathbf{R} \times V$ open!), and for $x \neq 0$ $[x, h(x)] \notin \mathbf{R} \times V$, which completes the proof of the Theorem.

3. Proof of the Brown's Theorem

As we have mentioned in the Introduction, it is the Theorem 2 in [1] which has a key importance for our result. We use this opportunity to present here our proof of the Brown's Theorem 2. Although we follow the main idea of Brown's proof, we tried to simplify it (and to avoid partial slight inexactness or inconsequentness of some steps of the original proof).

We use only the following well-known (see e. g. [4], [5]) properties of Baire class 1 functions:

(P1) if h is any Baire class 1 function and $P \subseteq \mathbf{R}$ is any nonempty perfect set, then there exists $c \in P$ such that $h \restriction P$ is continuous at c,

(P2) if h is any Baire class 1 function, then the set of all points of continuity of h is dense in $\mathbf{R}$.

(Actually, for functions $h : \mathbf{R} \to \mathbf{R}$, (P1) holds as the equivalent characterization of Baire class 1 functions. The conclusion of (P1) is also used as the definition of the notion of barely continuous function (see e. g. [5]). It follows from [5], Theorem 3.1, that the conclusion of (P2) holds for all barely continuous functions in metric spaces, in general.)

Proof of [1], Theorem 2. We use the symbols Cl for the closure in the plane, U_x for an open square in G with the centre $[x, h(x)]$ and Π_x for the projection to the x-axis.

Let h be any Baire class 1 function which is not almost continuous. We prove that h doesn't satisfy (E). Let G be an open set containing h, but containing no continuous function. Put

$$G_1 = \{X \in \mathbb{R} \times \mathbb{R};\ (\exists g \subseteq G)$$
$$(g \text{ is a continuous function with } \operatorname{dom} g, \operatorname{rng} g \subseteq \mathbb{R} \ \&\ 0 \in \operatorname{dom} g \ \&\ X \in g)\},$$
$$G_2 = G - \mathrm{Cl}(G_1),\ A = \{x \in \mathbb{R};\ [x, h(x)] \in G_1\},\ B = \{x \in \mathbb{R}; [x, h(x)] \in G_2\},$$
$$K = \mathbb{R} - (\operatorname{int}(A) \cup \operatorname{int}(B)).$$

Let M be the set of all isolated points of K and put $P = K - M$. (It can be easily shown that G_1 is open, but we don't use this fact in our proof.) We use the Propositions 1, 2, 3 and the Observation in the rest of our proof.

Proposition 1. *Let (a, b) and (c, d) be intervals.*

(i) *Let for $b, c \geq 0$ $(a, b) \subseteq A$ and $(c, d) \subseteq B$ hold. Then*
($b \notin A \Rightarrow h$ doesn't satisfy (E)) and ($c \notin B \Rightarrow h$ doesn't satisfy (E)).

(ii) *Let for $a, d \leq 0$ $(a, b) \subseteq A$ and $(c, d) \subseteq B$ hold. Then*
($a \notin A \Rightarrow h$ doesn't satisfy (E)) and ($d \notin B \Rightarrow h$ doesn't satisfy (E)).

Proof. We show (i); (ii) is analogical. Let $b \notin A$, i.e. $[b, h(b)] \notin G_1$ and consider U_b. If there exists $[x, h(x)] \in U_b$ with $x \in (a, b)$, then $[b, h(b)]$ must belong to G_1, too ($[x, h(x)]$ can be easily connected with $[b, h(b)]$ by a continuous function inside $U_b \subseteq G$) - a contradiction. Therefore h fails (E) at b. Similarly, let $c \notin B$, i.e. $[c, h(c)] \notin G_2$, i.e. $[c, h(c)] \in \mathrm{Cl}(G_1)$ and consider U_c small enough (not "overflowing" the right bound d). U_c contains a point $[w, y] \in G_1$ with $w < d$. If we had $[x, h(x)] \in U_w$ for some $x \in (w, d)$, then $[x, h(x)]$ had to belong to G_1, too ($[w, y]$ can be easily connected with $[x, h(x)]$ by a continuous function inside $U_w \subseteq G$) - a contradiction. Therefore h fails (E) at w.

Proposition 2. $\operatorname{int}(A) \cup \operatorname{int}(B)$ *is dense in* $\mathbb{R}$.

Proof. Take any open interval J. h has a point of continuity $c \in J$, $c \geq 0$ (the case $c < 0$ is analogical). Consider the square U_c such that $\Pi_x(U_c) = (c - \varepsilon, c + \varepsilon) \subseteq J$ and for all $x \in \Pi_x(U_c)$, $[x, h(x)] \in U_c$. If $[c, h(c)] \in G_1$, then $(c, c+\varepsilon) \subseteq J \cap \operatorname{int}(A)$, if $[c, h(c)] \notin G_1$, then $(c - \varepsilon, c) \subseteq J \cap \operatorname{int}(B)$.

Observation. *If any left endpoint $c \geq 0$ (right endpoint $d \leq 0$) of some component of B is an isolated point of K (i.e. an element of M), then h doesn't satisfy (E). If any right endpoint $b \geq 0$ (left endpoint $a \leq 0$) of some component of A is an isolated point of K (i.e. an element of M), then h doesn't satisfy (E).*

Proof. Follows immediately from the Proposition 1.

Proposition 3. *If P has an isolated point, then h doesn't satisfy (E).*

Proof. Take an isolated point $c \in P$ and a sequence $\{c_n\}_{n\in\mathbb{N}} \to c$, $c_n \in M$.
Case (i). For each $c_n \geq 0$ there exist components A_n of A, B_n of B such that $\sup B_n = c_n = \inf A_n$. Put $d_n = \inf B_n$ or $d_n = \sup A_n$ such that $|c-d_n| \leq |c-c_n|$. (Similarly for $c_n < 0$ we have A_n, B_n such that $\inf A_n = c_n = \sup B_n$ and put $d_n = \sup B_n$ or $d_n = \inf A_n$.) Clearly $d_n \to c$. Since c is an isolated point of P, there must exist $d_n \in K - P = M$. Then such d_n satisfies one of the assumptions of the Observation.
Case (ii). Since all c_n's are isolated points of K, there must exist $c_n \geq 0$ which is a left endpoint of a component of B and a right endpoint of a component of A (the case $c_n < 0$ is analogical). Again the rest is the Observation.

Finally we show that the only remaining case:

P perfect,
P contains all left endpoints $c \geq 0$, all right endpoints $d \leq 0$ of components of B and all right endpoints $b \geq 0$, all left endpoints $a \leq 0$ of components of A

is impossible. Since P is perfect nonempty and h is a Baire class 1 function, $h \restriction P$ is continuous at some $c \in P$, $c \geq 0$ (the case $c < 0$ is analogical). Consider the square U_c such that for all $x \in P \cap \Pi_x(U_c)$, $[x, h(x)] \in U_c$. Since c is not an isolated point of P, we have infinitely many components of A and of B in $\Pi_x(U_c)$, and for all left endpoints x of these components of B and right endpoints x of these components of A we obtain $x \in P \cap \Pi_x(U_c)$. Therefore for all such x, $[x, h(x)] \in U_c \subseteq G$, which leads to a contradiction completing the proof of Brown's Theorem.

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ZBORNÍK VEDECKÝCH PRÁC
Z MEDZINÁRODNEJ KONFERENCIE

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1. sekcia

MATEMATIKA A JEJ APLIKÁCIE V TECHNICKÝCH VEDÁCH

Autor	:	Kolektív
Editor	:	Strojnícka fakulta TU v Košiciach
Vydavateľ	:	Vydavateľstvo STROFEK
Vydanie	:	Prvé
Náklad	:	80
Rozsah	:	144 strán
Formát	:	A5
Tlač	:	Tlačiareň STROFEK

Košice, september 1997

ISBN 80-967636-7-9