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NOVÉ TRENDY V STROJÁRSTVE NA PRAHU TRETIEHO TISÍCROČIA

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MATEMATIKA A JEJ APLIKÁCIE V TECHNICKÝCH VEDÁCH

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Obsah

On simple continuity

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Let X, Y be two topological spaces. Let $\mathcal{F}$ be a class of functions $f : X \to Y$. A subset A of X with the property that whenever $f \in \mathcal{F}$ is constant on A, then f must be constant on X, is said to be a *stationary set* for $\mathcal{F}$. Denote by $\mathfrak{S}(\mathcal{F})$ the set of all stationary sets for the class $\mathcal{F}$. Observe that if $\mathcal{F}_1 \subset \mathcal{F}_2$, then $\mathfrak{S}(\mathcal{F}_2) \subset \mathfrak{S}(\mathcal{F}_1)$. (See [2], p.199.)

In this paper we give a complete characterization of stationary sets for the class of simply continuous functions.

A subset A of a topological space X will be said to be *simply open* if $A = D \cup E$ where D is open and E is nowhere dense in X. Note that a set A is simply open if and only if $\operatorname{Fr} A$ (where $\operatorname{Fr} A = \operatorname{Cl} A - \operatorname{Int} A$) is nowhere dense.Let X, Y be two topological spaces. A function $f : X \to Y$ is called simply continuous if for every open U in Y, the set $f^{-1}(U)$ is simply open in X. In the sequel $SC(X,Y)$ denotes the set of all simply continuous functions $f : X \to Y$. (See [1].)

Stationary sets for the class of simply continuous functions

We suppose that Y is a T_1-space which has at least two elements for the whole this paragraph.

Theorem 1. *Let X be a nonempty topological space. Let A be a subset of X. Then $A \in \mathfrak{S}(SC(X,Y))$ if and only if the set $X - A$ has not a nonempty simply open subset.*

Proof. Necessity. Deny. Suppose that there exists a nonempty simply open set $B \subset X$ such that $B \subset X - A$. Choose $u, v \in Y$ such that $u \neq v$. Define the function $f : X \to Y$ as follows

$$f(x) = \begin{cases} u & \text{for } x \in B, \\ v & \text{otherwise.} \end{cases}$$

Then $f \in SC(X,Y)$ and f is constant on A. But f is not constant on X.

Sufficiency. By contradiction. Suppose that $X - A$ has not a nonempty simply open subset. Let $f \in SC(X,Y)$, $f(x) = c$ for $x \in A$, $f(b) \neq c$ for some $b \in X - A$. Let U be an open neighbourhood of the point $f(b)$ such that $c \notin U$. Since $f^{-1}(U)$ is nonempty simply open in X, by the assumption we have $A \cap f^{-1}(U) \neq \emptyset$. Thus $c \in U$, a contradiction.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Corollary 1. *Let X be a nonempty topological space. Then every stationary set for the class $SC(X,Y)$ is dense in X.*

Corollary 2. *Let X be a nonempty topological space. Then $\mathfrak{S}(SC(X,Y)) = \{X\}$ if and only if every singleton (in X) is simply open.*

Definition. A topological space X is a $T_{\frac{1}{2}}$-space if and only if $\mathrm{Cl}(\{a\}) - \{a\}$ is closed for each $a \in X$ (see [5]).

Theorem 2. *Let X be a nonempty $T_{\frac{1}{2}}$-space. Then $\mathfrak{S}(SC(X,Y)) = \{X\}$.*

Proof. Let $a \in X$. Suppose that $\{a\}$ is not open. We shall prove that $\{a\}$ is nowhere dense in X. Since X is a $T_{\frac{1}{2}}$-space, the set $W = \{a\} \cap \mathrm{Int}\,\mathrm{Cl}\{a\}$ is open. Then $W = \mathrm{Int}\,W = \mathrm{Int}\{a\} = \emptyset$. Hence $a \notin \mathrm{Int}\,\mathrm{Cl}\{a\}$, therefore $\mathrm{Int}\,\mathrm{Cl}\{a\} = \emptyset$. By Corollary 2 the desired property is ensured.

The following example shows that the assumption "$T_{\frac{1}{2}}$-space" in Theorem 2 cannot be replaced by the assumption "T_0-space".

Example. Let X be an infinite set, $a \in X$. Define the open sets of a topology τ on X to be $\emptyset$ and all sets of the form $A \cup \{a\}$ where $A \subset X$, $X - A$ is finite. Then (X,τ) is a T_0-space, but $X - \{a\} \in \mathfrak{S}(SC(X,Y))$.

Stationary sets for some other generalizations of continuity

A subset A of a topological space X will be said to be ***pseudo-open*** if $A = G \cup H$ where G is open and H is a set of the first Baire category.

Let X, Y be two topological spaces. A function $f : X \to Y$ is called ***pseudo-continuous*** if for every open U in Y, the set $f^{-1}(U)$ is pseudo-open in X. (See [4].)

Let X be a topological and Y a metric spaces (with the metric d). A function $f : X \to Y$ is said to be ***cliquish*** at the point $p \in X$ if for each neighbourhood $U(p)$ of the point p and each $\varepsilon > 0$ there exists a nonempty open set $U \subset U(p)$ such that $d(f(x), f(y)) < \varepsilon$ holds for each two points $x, y \in U$. A function $f : X \to Y$ is said to be cliquish on X if it is cliquish at each point $a \in X$. (See [6].)

Let X, Y be two topological spaces, let $f : X \to Y$. Denote by C_f (D_f) the set of all continuity (discontinuity) points of the function f. A function $f : X \to Y$ is called ***pointwise discontinuous*** if C_f is dense in X.

In the sequel $PC(X,Y)$, $Q^*(X,Y)$, and $PD(X,Y)$ denote the sets of all functions $f : X \to Y$ which are pseudo-continuous, cliquishon X, and pointwise discontinuous, respectively.

Theorem 3. *Let X be a nonempty topological space and Y a metric space which has at least two elements. Then*

$$\mathfrak{S}(PC(X,Y)) = \mathfrak{S}(Q^*(X,Y)) = \mathfrak{S}(PD(X,Y)) = \mathfrak{S}(SC(X,Y)).$$

Proof. Since $PD(X,Y) \subset Q^*(X,Y) \subset PC(X,Y)$ (see [3] and [4]), we have

$$\mathfrak{S}(PC(X,Y)) \subset \mathfrak{S}(Q^*(X,Y)) \subset \mathfrak{S}(PD(X,Y)).$$

Now, we shall prove that $\mathfrak{S}(PD(X,Y)) \subset \mathfrak{S}(SC(X,Y))$. Let $A \subset X$, $A \notin \mathfrak{S}(SC(X,Y))$. Then by Theorem 1, there exists a nonempty simply open set $S \subset X$ such that $S \subset X - A$. Choose $u, v \in Y$ such that $u \neq v$. Define the function $f : X \to Y$ as follows

$$f(x) = \begin{cases} u & \text{for } x \in S, \\ v & \text{otherwise.} \end{cases}$$

Since $D_f \subset \operatorname{Fr} S$, the set D_f is nowhere dense in X. Then $f \in PD(X,Y)$, f is constant on A, but f is not constant on X. Thus $a \notin \mathfrak{S}(PD(X,Y))$.
Finally, we shall prove that $\mathfrak{S}(SC(X,Y)) \subset \mathfrak{S}(PC(X,Y))$. Let $A \subset X$, $A \notin \mathfrak{S}(PC(X,Y))$. Then there exists $g \in PC(X,Y)$ such that $g(x) = c$ for some $x \in A$, $g(b) \neq c$ for some $b \in X - A$. Thus the set $X - g^{-1}(c) = g^{-1}(Y - \{c\})$ is nonempty pseudo-open. Choose a nonempty simply open set $B \subset X$ such that $B \subset X - g^{-1}(c)$. Choose $a \in Y$ such that $a \neq c$. Define the function $h : X \to Y$ as follows

$$h(x) = \begin{cases} a & \text{for } x \in B, \\ c & \text{otherwise.} \end{cases}$$

Then $h \in \mathfrak{S}(SC(X,Y))$, h isconstant on A, but h is not constant on X. Therefore $A \notin \mathfrak{S}(SC(X,Y))$.

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