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When distance means money

by J. DOBOŠ
Technicka Univerzita, Katedra Matematiky SjF, Švermova 9, 04001 Košice, Slovakia

Z. PIOTROWSKI
Department of Mathematics and Statistics, Youngstown State University, Youngstown, OH 44555-0001, USA

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In this paper, we offer a natural way of acquainting the student with the notion of the distance function. Real-life situations and graphs of 'easy-to-see' step functions should appeal to students' intuition and make the topic of distance-preserving functions easily accessible.

We begin with three examples:

Example 1. You are in New York City, just got a cab—you get in. The meter shows \$1·50.

This is a flat rate, as the driver says. Now every $\frac{1}{8}$ of a mile will cost you \$0·25. The graph is shown in Figure 1.

Example 2. Have you called your friend in Paris, France from NY? The current AT&T full rate is \$1·71 for the first minute and \$1·08 for every additional minute. This data is shown on the graph in Figure 2.

Example 3. Imagine that you operate a truck fleet. Observe that the distance travelled by a truck corresponds to a certain amount of money spent on fuel. Conservative estimates show about 10 miles per gallon. Assume 1 gallon of fuel costs \$1·00.

If we place the distance (in miles) on the x-axis and the cost (in \$) on the y-axis then the function $f: X \to Y$ is given by $f(x) = -[-(x/10)], x \geqslant 0$; where $[\cdot]$ denotes the greatest integer function.

What do these three examples have in common?

First, in each instance we replace the actual distance, or time, as in Example 2, by the cost.

Second, each function presented is a step function, i.e. f assumes only discrete sets of values; without loss of generality we may assume that the values of f are natural numbers.

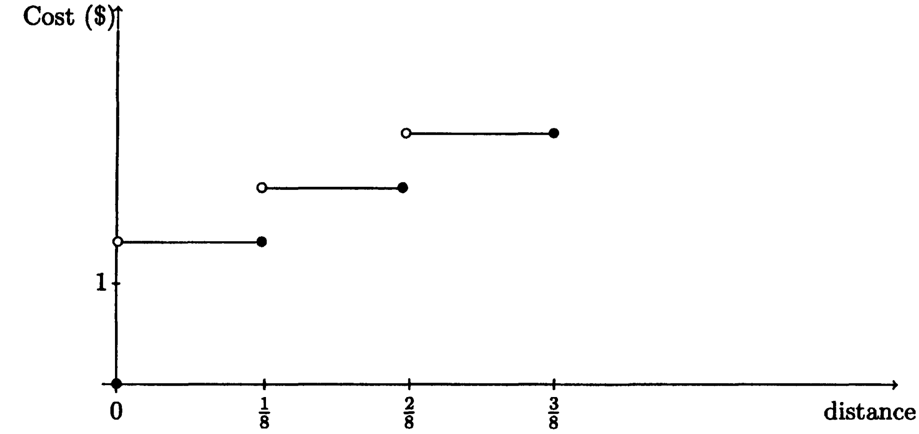

Figure 1

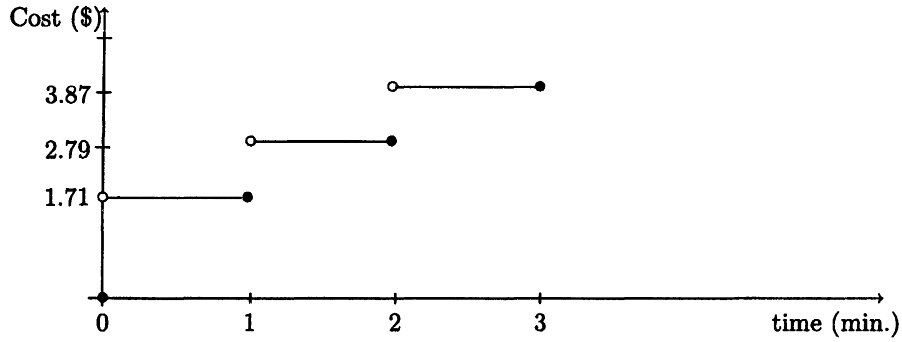

Figure 2

Let us recall, given X, a set of elements. We say that a function $d: X \times X \to R_+ \cup \{0\}$ is a *distance function*, or a *metric* , if the following axioms are met for any x, y and z in X.

Axiom 1: $d(x, y) = 0$ if and only if $x = y$
Axiom 2: $d(x, y) = d(y, x)$
Axiom 3: $d(x, y) \leqslant d(x, z) + d(z, y)$
The pair (X, d) is called a *metric space.*

Formalizing the above discussion we have the following general problem:
Given a distance d, we shall refer to d as 'the old metric', or 'the old distance'. We will consider these functions $f: [0, \infty) \to \mathbb{N} \cup \{0\}$ such that the composition ζ, defined by:

$$\zeta(x, y) = f(d(x, y)) \tag{i}$$

is a 'new' distance.

So, we will quantify the values of the old metrics by 'shuffling' the in-between numbers to nearby integer numbers. This, in turn, makes f a step function.

Probably the first example which comes to mind is the greatest integer function $[\cdot]$. But such a composition will not produce a metric. In fact, if $0 < d(x, y) < 1$, then $\zeta(x, y) = 0$ which in turn, implies $x = y$, which is a contradiction.

Slight modification, however, of the above example produces a suitable function, namely:

Lemma 1. Let $f_n : [0, \infty) \to \mathbb{N} \cup \{0\}$ be defined by $f_n(u) = -n[-u/n]$, $n = 1, 2, , \dots$ and $[\cdot]$ denotes the greatest integer function. Then f_n preserves the distance.

Proof. (1) Axiom 1:

$$\zeta_n(x, y) = 0 \Leftrightarrow x = y \Leftrightarrow -n\left[\frac{-d(x, y)}{n}\right] = 0 \Leftrightarrow -1 < \frac{d(x, y)}{n} \leqslant 0$$

However, since the values of d are nonnegative, we have: $-n < d(x, y) \leqslant 0 \wedge d(x, y) \geqslant 0$ which is equivalent to

$$\zeta_n(x, y) = 0 \Leftrightarrow d(x, y) = 0 \Leftrightarrow x = y$$

(2) Axiom 2: $\zeta_n(x, y) = \zeta_n(y, x)$. It follows immediately from the fact that $d(x, y) = d(y, x)$.

(3) Axiom 3: $\zeta_n(x, y) + \zeta_n(y, x) \geqslant \zeta_n(x, z)$. We have $d(x, y) + d(y, z) \geqslant d(x, z)$ hence

$$\frac{-d(x, y)}{n} + \frac{-d(y, z)}{n} \leqslant \frac{-d(x, z)}{n}$$

Now

$$\left[\frac{(-d(x, y)}{n} + \frac{-d(y, z)}{n}\right] \leqslant \left[\frac{-d(x, z)}{n}\right]$$

where $[\cdot]$ denotes the greatest integer value. Now, since $[a + b] \geqslant [a] + [b]$, we get

$$\left[\frac{-d(x, y)}{n}\right] + \left[\frac{-d(y, z)}{n}\right] \leqslant \left[\frac{-d(x, z)}{n}\right]$$

Therefore

$$\left(-n\left[\frac{-d(x, y)}{n}\right]\right) + \left(-n\left[\frac{-d(y, z)}{n}\right]\right) \geqslant -n\left[\frac{-d(x, z)}{n}\right]$$

which implies

$$\zeta_n(x, y) + \zeta_n(y, z) \geqslant \zeta_n(x, z)$$

There is yet another way to check whether ζ_n is a metric. Namely, following Kelley [1, p. 131].

Theorem 1. Let f be a real-valued function defined for nonnegative numbers, and such that f is continuous, (the continuity of f is needed only for the equivalence of d with e), nondecreasing and satisfying the following two conditions:

(*) $f(a) = 0 \Leftrightarrow a = 0$, and
(**) $f(a + b) \leqslant f(a) + f(b)$, for any a, b.

Let (Z, d) be a metric space and let $e(x, y) = f(d(x, y))$ for any $x, y \in X$. Then (X, e) is a metric space and the metrics d and e are equivalent.

Clearly the function $f : [0, +\infty) \to [0, +\infty)$ given by $f_n(u) = -n[-(u/n)]$, $n = 1, 2 \dots$ is nondecreasing and also (*) and (**) hold for f_n and any nonnegative a and b.

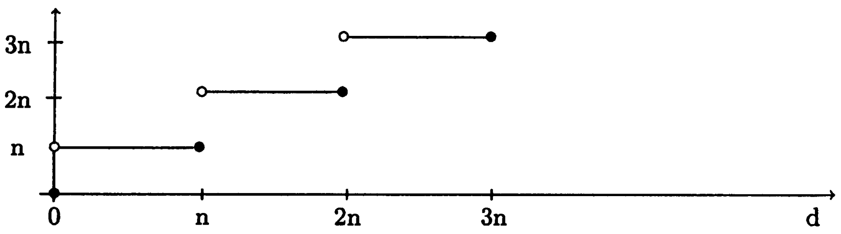

Figure 3

The monotonicity part can be easily checked, whereas subadditivity (condition (**)) of f_n is met due to the following routine estimate:

$$f_n(a+b) = -n\left[\frac{-(a+b)}{n}\right] = -n\left[\left(\frac{-a}{n}\right) + \left(\frac{-b}{n}\right)\right] \leqslant \left(-n\left[-\frac{a}{n}\right]\right) + \left(-n\left[-\frac{b}{n}\right]\right)$$

$$= f_n(a) + f_n(b)$$

You have noticed, in example 3, that assuming 10 miles as a unit segment on the horizontal axis, f_1 defines a 'new distance' – the cost.

Observe Examples 1 and 2 (see Figures 1 and 2); we cannot use either one of f_n, $n = 1, 2, \ldots$, due to the fact, that the 'steps' in the step functions shown in Figures 1 and 2, are uneven.

We need a deeper, more sophisticated analysis of what f must be in order to be distance-preserving. Let us begin with the following:

Lemma 2. Assume $f : [0, \infty) \to \mathbb{R}$ be nondecreasing. Then $\frac{(f(a) + f(b))}{2} \leqslant f(a+b)$.

Proof. We have $f(a) \leqslant f(a+b)$ and $f(b) \leqslant f(a+b)$ so, adding both inequalities (sideways) we obtain

$$f(a) + f(b) \leqslant 2f(a+b)$$

which gives

$$\frac{f(a) + f(b)}{2} \leqslant f(a+b)$$

□

Corollary. Assume that $f : [0, \infty) \to \mathbb{R}$ is nondecreasing and subadditive. Then:

$$\frac{f(a) + f(b)}{2} \leqslant f(a+b) \leqslant f(a) + f(b)$$

Let us now start with a somewhat general case. Define $g : [0, \infty) \to \mathbb{N} \cup \{0\}$ as shown in Figure 4.

Now, how to define g for $x > 1$? By the above Corollary

$$a = \frac{g(1) + g(1)}{2} \leqslant g(2) = g(1+1) \leqslant g(1) + g(1) = 2a$$

hence $a \leqslant g(2) \leqslant 2a$.

Yet a better estimate can be obtained by using theorem 2, below. In order to

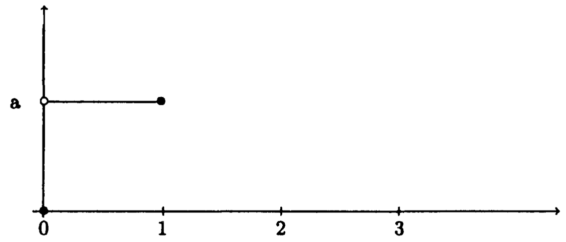

Figure 4

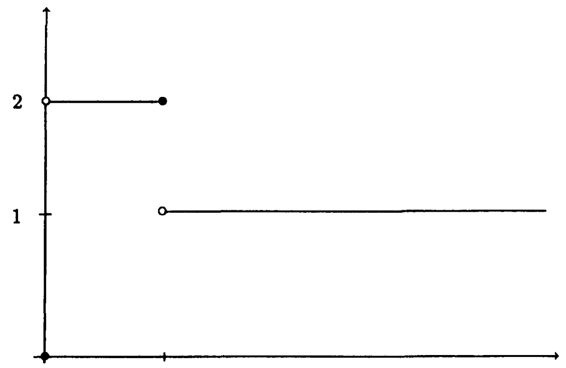

Figure 5

decide which step functions are distance-preserving we need to examine the necessity of the assumptions in Theorem 1. You will find [2] that if f is distance-preserving, then f is subadditive.

Readers will probably be surprised to learn that a distance-preserving function does not have to be nondecreasing. An appropriate example will follow from Theorem 2. We first need

Definition [2]. Let a, b and c be positive numbers. We say that the triplet (a, b, c) is a *triangle triplet* iff $a \leqslant b + c$, $b \leqslant a + c$ and $c \leqslant a + b$.

Theorem 2 [2]. Let $f : [0, \infty) \to [0, \infty)$. Then f is distance preserving if and only if:

(a) f vanishes exactly at the origin, and

(b) if (a, b, c) is a triangle triplet, then so is $(f(a), f(b), f(c))$.

Based on the above criteria the function f given in Figure 5 is distance-preserving (!), even though it is not nondecreasing.

Observe that Theorem 2 can also be applied in the following example.

Example 4. Let the function h be given by

$$h(x) = \begin{cases} 0 & \text{if } x = 0 \\ 3 & \text{if } 0 < x \leqslant 1 \\ 1 & \text{if } x > 1 \end{cases}$$

h is not distance-preserving. In fact $(1, 2, 3)$ is a triangle triplet, while $(f(1), f(2), f(3))$ is not:

$$3 = f(1) \nleqslant f(2) = f(3) = 1 + 1 = 2$$

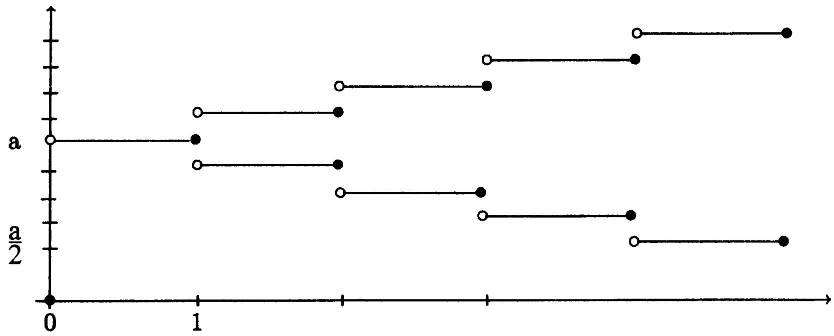

Figure 6

Applying theorem 2 and assuming $\lim_{x\to 0^+} f(x) = a$ and $f(x) = a$ for $x \in (0, 1)$, one can present a region in the plane, where the graph of any distance-preserving, integer-valued function f is contained (Figure 6).

The general case, when the initial interval $(0, 1)$ on the horizontal axis is replaced by an arbitrary $(0, b)$ is somewhat similar, although not identical.

We hope that the reader will find it interesting to determine the above mentioned region for specific as and bs.

References

[1] Kelley, J. L., *General topology* (D. van Nostrand Co., New York), 1955.
[2] Borsik, J., and Doboš, J., 1981, Functions whose composition with every metric is a metric, *Math. Slovaca*, **31**, 3–12 (in Russian).