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ON MODIFICATIONS OF THE EUCLIDEAN METRIC ON REALS

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ABSTRACT. The paper is concerned with some differentiable properties of Euclidean metric preserving functions.

Denote by $\mathcal{O}$ the set of all functions $f\colon [0,+\infty) \to [0,+\infty)$ with $f(0)=0$. Let (M,d) be a metric space. For each $f \in \mathcal{O}$ define the function $d_f\colon M \times M \to [0,+\infty)$ as follows

$$d_f(x,y) = f\bigl(d(x,y)\bigr) \qquad \text{for each} \quad x,y \in M\,.$$

Denote by $\mathcal{M}$ the set of all functions $f \in \mathcal{O}$ such that for each metric space (M,d) the function d_f is a metric on M. Denote by $\mathcal{M}_0$ $(\mathcal{M}_1)$ the set of all functions $f \in \mathcal{O}$ such that e_f is a pseudometric (metric) on the real line, where $e\colon \mathbb{R} \times \mathbb{R} \longrightarrow [0,+\infty)$ is the Euclidean metric on $\mathbb{R}$ (i.e., $e(x,y) = |x-y|$ for each $x,y \in \mathbb{R}$).

Let a, b and c be positive numbers. We say that the triplet (a,b,c) is triangle iff $a \leqq b+c$, $b \leqq a+c$ and $c \leqq a+b$. It is known (see [1]) that $f \in \mathcal{M}$ iff f vanishes exactly at the origin and maps each triangle triplet (a,b,c) to a triangle triplet.

PROPOSITION 1. *Let $f \in \mathcal{O}$. Then*

a) *$f \in \mathcal{M}_0$ iff f maps each triangle triplet $(a,b,a+b)$ to a triangle triplet;*

b) *$f \in \mathcal{M}_1$ iff $f \in \mathcal{M}_0$ and f vanishes exactly at the origin.*

Denote by F the even extension of $f \in \mathcal{O}$, i.e., $F\colon \mathbb{R} \to [0,+\infty)$, $F(x) = f(|x|)$ for each $x \in \mathbb{R}$. It is not difficult to prove

PROPOSITION 2. *Let $f \in \mathcal{O}$. Then the following assertions are equivalent*

(i) $f \in \mathcal{M}_0$,

(ii) *F is subadditive,*

(iii) $\forall x,y \in [0,+\infty)\colon |f(x)-f(y)| \leqq f(|x-y|)$.

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COROLLARY 1. *Let $f \in \mathcal{M}_0$. Then f is continuous iff it is continuous at the origin.*

PROPOSITION 3. (Compare to [3].) *Let $f \in \mathcal{M}_0$, $t > 0$. Then F is periodic with the period t iff $f(t) = 0$.*

Proof. Suppose that $f(t) = 0$. Let $x \in \mathbb{R}$. Then $F(x+t) \leqq F(x) + F(t) = F(x) \leqq F(x+t) + F(-t) = F(x+t)$, which yields $F(x+t) = F(x)$. □

COROLLARY 2. *Let $f \in \mathcal{M}_0$. Suppose that f is differentiable on the right at some $s \in f^{-1}(0)$. Then for each $t \in f^{-1}(0)$ there exist both one-sided derivatives of F and $F'_+(t) = -F'_-(t) = f'_+(s)$.*

PROPOSITION 4. *Let $f \in \mathcal{M}_0$, $t \in F^{-1}(0)$. Then F is differentiable at t iff F is constant.*

Proof. By Corollary 2 F is differentiable at the origin and $F'(0) = 0$. Let $a > 0$. We shall show that $f(a) = 0$. Let $\varepsilon > 0$. Then there is $\delta > 0$ such that for each $x \in (0, \delta)$ we have $\frac{f(x)}{x} < \frac{\varepsilon}{a}$. Choose $n \in \mathbb{N}$ such that $\frac{a}{n} < \delta$. Then $f(\frac{a}{n}) < \frac{\varepsilon}{n}$. Therefore $f(a) \leqq n \cdot f(\frac{a}{n}) < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we obtain $f(a) = 0$. □

COROLLARY 3. *Let $f \in \mathcal{M}_0$ be nonconstant. If f is differentiable on $(0, +\infty)$, then $f \in \mathcal{M}_1$.*

It is known that for each $f \in \mathcal{M}$ we have

$$\forall a, b \geqq 0,\ a \leqq 2b\colon\ f(a) \leqq 2f(b)\,. \tag{$*$}$$

(See [1].) The following example shows that for functions $f \in \mathcal{M}_1$ this assertion is not true.

EXAMPLE 1. Define $f\colon [0, +\infty) \to [0, +\infty)$ by $f(x) = |\sin x| + |\sin \sqrt{2}x|$ for each $x \geqq 0$. Then $f \in \mathcal{M}_1$, but $\liminf\limits_{x \to +\infty} f(x) = 0$.

Now we will describe a construction of differentiable functions $f \in \mathcal{M}_1$ with $\liminf\limits_{x \to +\infty} f(x) = 0$.

LEMMA 1. *Let $f \in \mathcal{M}$, $n \in \mathbb{N}$. Define $f_n\colon [0, +\infty) \to [0, +\infty)$ as follows*

$$f_n(x) = \begin{cases} f(x)\,, & x \in [0, 2^{n-1}]\,, \\ f(2^n - x)\,, & x \in (2^{n-1}, 2^n]\,, \\ f_n(x - k \cdot 2^n)\,, & x \in (k \cdot 2^n, (k+1) \cdot 2^n]\,, \quad (k = 1, 2, \dots)\,. \end{cases}$$

Then $f_n \in \mathcal{M}_0$.

Proof. Let $x, y > 0$. Then there are $k, l \in \{0, 1, 2, 3, \dots\}$ such that $k \cdot 2^n < x \leqq (k+1) \cdot 2^n$ and $l \cdot 2^n < y \leqq (l+1) \cdot 2^n$. Put $a = x - k \cdot 2^n$ and $b = y - l \cdot 2^n$. Evidently $a, b \in (0, 2^n]$. Suppose that $a \leqq b$. We distinguish six cases.

1) Let $a, b, a+b \in (0, 2^{n-1}]$. Since $(a, b, a+b)$ is a triangle triplet, $\big(f_n(x), f_n(y), f_n(x+y)\big) = \big(f(a), f(b), f(a+b)\big)$ is a triangle triplet.
2) Let $a, b \in (0, 2^{n-1}]$, and $a+b \in (2^{n-1}, 2^n]$. Since $(a, b, 2^n - a - b)$ is a triangle triplet, $\big(f_n(x), f_n(y), f_n(x+y)\big) = \big(f(a), f(b), f(2^n - a - b)\big)$ is a triangle triplet.
3) Let $a \in (0, 2^{n-1}]$, and $b, a+b \in (2^{n-1}, 2^n]$. Since $(a, 2^n - b, 2^n - a - b)$ is a triangle triplet, $\big(f_n(x), f_n(y), f_n(x+y)\big) = \big(f(a), f(2^n - b), f(2^n - a - b)\big)$ is a triangle triplet.
4) Let $a \in (0, 2^{n-1}]$, $b \in (2^{n-1}, 2^n]$, and $a+b \in (2^n, 3 \cdot 2^{n-1}]$. Since $(a, 2^n - b, a+b-2^n)$ is a triangle triplet, $\big(f_n(x), f_n(y), f_n(x+y)\big) = \big(f(a), f(2^n - b), f(a+b-2^n)\big)$ is a triangle triplet.
5) Let $a, b \in (2^{n-1}, 2^n]$ and $a+b \in (2^n, 3 \cdot 2^{n-1}]$. Since $(2^n - a, 2^n - b, a+b-2^n)$ is a triangle triplet, $\big(f_n(x), f_n(y), f_n(x+y)\big) = \big(f(2^n - a), f(2^n - b), f(a+b-2^n)\big)$ is a triangle triplet.
6) Let $a, b \in (2^{n-1}, 2^n]$, and $a+b \in (3 \cdot 2^{n-1}, 2^{n+1}]$. Since $(2^n - a, 2^n - b, 2^{n+1} - a - b)$ is a triangle triplet, $\big(f_n(x), f_n(y), f_n(x+y)\big) = \big(f(2^n - a), f(2^n - b), f(2^{n+1} - a - b)\big)$ is a triangle triplet.

□

As a corollary we obtain

THEOREM. *Let $f \in \mathcal{M}$. Suppose that $f(x) = 1$ for each $x \geqq 1$. Define $f_0\colon [0, +\infty) \to [0, +\infty)$ as follows*

$$f_0(x) = \sup\{2^{1-n} \cdot f_n(x);\ n \in \mathbb{N}\} \quad \text{for each} \quad x \geqq 0.$$

Then $f_0 \in \mathcal{M}_1$ and $f_0(2^n) = 2^{-n}$ for each $n \in \mathbb{N}$.

The following example shows that there is a differentiable function $f \in \mathcal{M}_1$ with $\liminf\limits_{x \to +\infty} f(x) = 0$.

EXAMPLE 2. Let $f \in \mathcal{O}$ be such that

(1) f is nondecreasing,

(2) f is differentiable on $[0, +\infty)$,

(3) $f(a_n) = 2^{1-n}$ $(n \in \mathbb{N})$,

(4) $f'(a_n) = 0$ $(n \in \mathbb{N})$,

(5) $f(x) = 1$ for each $x \geqq 1$,

(6) $f(x) \geqq k_n \cdot x$ for each $x \in (a_{n+1}, a_n)$ $(n \in \mathbb{N})$,

(7) $f'(x) \leqq k_{n+1}$ for each $x \in (a_{n+1}, a_n)$ $(n \in \mathbb{N})$,

where $a_n = \frac{n+1}{n \cdot 2^n}$ and $k_n = \frac{2^{1-n}}{a_n}$ $(n \in \mathbb{N})$.

Since $f = \sup_n g_n$, where $g_n \colon [0, +\infty) \to [0, +\infty)$,

$$g_n(x) = \begin{cases} k_{n+1} \cdot x, & x \in [0, a_{n+1}), \\ f(x), & x \in [a_{n+1}, a_n], \\ 2^{1-n}, & x \in (a_n, +\infty), \end{cases}$$

$(n \in \mathbb{N})$, we have $f \in \mathcal{M}$. By Theorem $f_0 \in \mathcal{M}_1$ and $\liminf_{x \to +\infty} f_0(x) = 0$. It is not difficult to verify that f_0 is differentiable on $[0, +\infty)$.

By this method it is not difficult to construct a singular function $f \in \mathcal{M}_1$ with $\liminf_{x \to +\infty} f(x) = 0$.

EXAMPLE 3. Let $c \colon [0, 1] \to [0, 1]$ be the standard Cantor function. (See [2].) Define $f \colon [0, +\infty) \to [0, +\infty)$ as follows

$$f(x) = \begin{cases} c(x), & x \in [0, 1], \\ 1, & \text{otherwise.} \end{cases}$$

It is not difficult to verify that $f \in \mathcal{M}$. By Theorem we obtain $f_0 \in \mathcal{M}_1$ and $\liminf_{x \to +\infty} f_0(x) = 0$. It is easy to see that f_0 is singular.

Note that if $f \in \mathcal{M}$ is continuous, f_0 is almost periodic. In this connection a question arises of whether every continuous function $f \in \mathcal{M}_1$ with $\liminf_{x \to +\infty} f(x) = 0$ is almost periodic.

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