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THE STANDARD CANTOR FUNCTION IS SUBADDITIVE

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ABSTRACT. In this paper the subadditivity of the Cantor function $\phi\colon [0,1] \to [0,1]$ is proved.

Let us begin by recalling that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is said to be subadditive, if it satisfies the inequality $f(x+y) \leqq f(x) + f(y)$ whenever $x, y \in \mathbb{R}$. (See [2] and [4].)

The usual definition of the standard Cantor function involves the classic middle-thirds description of the standard Cantor set. (See [1] and [3].) We offer an alternate definition of this function.

Define a sequence of functions $\phi_n : \mathbb{R} \longrightarrow [0,1]$ by

$$\phi_0(x) = \begin{cases} 0 & \text{if } x \le 0, \\ x & \text{if } 0 \le x \le 1, \\ 1 & \text{if } x \ge 1, \end{cases} \quad \phi_{n+1}(x) = \begin{cases} \frac{1}{2} \cdot \phi_n(3x) & \text{if } x \le \frac{2}{3}, \\ \frac{1}{2} + \frac{1}{2} \cdot \phi_n(3x-2) & \text{if } x \ge \frac{1}{3}. \end{cases}$$

It is easy to check that each ϕ_n is non-decreasing, that $\phi_n(x) = 0$ for all $x \le 0$, that $\phi_n(x) = 1$ for all $x \ge 1$, and that the two lines in the definition of ϕ_{n+1} agree in the overlap of their domains, both giving $\phi_{n+1}(x) = \frac{1}{2}$ when $\frac{1}{3} \le x \le \frac{2}{3}$.

Put $\phi = \lim_{n \to +\infty} \phi_n$. It is not difficult to verify that the restriction of ϕ to $[0,1]$ is the standard Cantor function. The functions ϕ_n are polygonal approximations of ϕ.

Theorem. *The standard Cantor function is subadditive.*

Proof. The function ϕ is the pointwise limit of the functions ϕ_n as $n \to +\infty$. So to prove the subadditivity of ϕ, it suffices to prove the subadditivity of all ϕ_n, which we do by induction on n. The case $n = 0$ is trivial, so we proceed to the induction step from n to $n+1$. Let $x, y \in \mathbb{R}$, $x \ge y$. Here we consider several cases.

Case 1: $y \le 0$. This case is trivial as f_{n+1} is monotone.

Case 2: $y \ge \frac{1}{3}$. In this case,

$$\phi_{n+1}(x+y) \le 1 = \tfrac{1}{2} + \tfrac{1}{2} \le \phi_{n+1}(x) + \phi_{n+1}(y).$$

Case 3: $x \le \frac{1}{3}$. As x, y and $x+y$ are all $\le \frac{2}{3}$, we have

$$\begin{aligned} \phi_{n+1}(x+y) &= \tfrac{1}{2} \cdot \phi_n(3x+3y) \\ &\le \tfrac{1}{2} \cdot \phi_n(3x) + \tfrac{1}{2} \cdot \phi_n(3y) = \phi_{n+1}(x) + \phi_{n+1}(y). \end{aligned}$$

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Case 4: $0 \le y \le \frac{1}{3} \le x$. As $x + y \ge \frac{1}{3}$, we have

$$\begin{aligned}\phi_{n+1}(x+y) &= \tfrac{1}{2} + \tfrac{1}{2} \cdot \phi_n(3x+3y-2) \\ &\le \tfrac{1}{2} + \tfrac{1}{2} \cdot \phi_n(3x-2) + \tfrac{1}{2} \cdot \phi_n(3y) = \phi_{n+1}(x) + \phi_{n+1}(y).\end{aligned}$$

These four cases exhaust all the possibilities, so the proof is complete. □

Let us recall that a modulus of continuity is a function f defined, continuous, nondecreasing and subadditive on $[0, 1]$ with $f(0) = 0$.

Corollary. *The standard Cantor function is a modulus of continuity.*

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