Internat. J. Math. & Math. Sci.
VOL. 19 NO. 1 (1996) 199-200

A NOTE ON METRIC PRESERVING FUNCTIONS

JOZEF DOBOS and ZBIGNIEW PIOTROWSKI

Katedra Matematiky SjF, Technická Univerzita,
Letná 9, 040 01 Košice, Slovak Republic
and
Department of Mathematics, Youngstown State University,
Youngstown OH 44555 USA

(Received March 15, 1993 and in revised form February 17, 1995)

ABSTRACT: The purpose of this note is to study modifications of the Euclidean metric on $\mathbb{R}$ with the following property: There is a monotone sequence of closed balls with empty intersection.

KEY WORDS AND PHRASES. Metric preserving functions.
1992 AMS SUBJECT CLASSIFICATION CODES. 26A30

1. INTRODUCTION:

DEFINITION 1. We call a function $f: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ metric preserving iff $f(d): M \times M \longrightarrow \mathbb{R}^+$ is a metric for every metric $d: M \times M \longrightarrow \mathbb{R}^+$, where (M, d) is an arbitrary metric space and $\mathbb{R}^+$ denotes the set of nonnegative reals. We denote by $\mathcal{M}$ the set of all metric preserving functions. (See Borsík [1], Borsík [2], Terpe [3].)

The following result is well known (see Borsík [1], Terpe [3]).

PROPOSITION 1. If $f: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a concave function vanishing exactly at origin then it is metric preserving.

It is well known that there is a complete metric space with the following property:

There is a monotone sequence of closed balls with empty intersection. (1.1)

In Juza [4] such a metric space (which is not discrete) has been constructed by a modification of the Euclidean metric on $\mathbb{R}$, where $\mathbb{R}$ denotes the set of reals

For each $f \in \mathcal{M}$ denote by d_f the metric on $\mathbb{R}$ defined as follows:

$$d_f(x, y) = f(|x-y|) \text{ for each } x, y \in \mathbb{R}.$$

We call d_f a *modification* of the Euclidean metric on $\mathbb{R}$. (See Terpe [3].)

EXAMPLE 1. Define $f: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ as follows:

$$f(x) = x, \text{ it } x \leq 2, f(x) = 1 + \frac{1}{x-1} \text{ if } x > 2.$$

In Juza [4] it is shown that $f \in \mathcal{M}$ and the metric space $(\mathbb{R}, d_f)$ has the property (1.1). The proof of (1.1) is based on the following property of the metric space $(\mathbb{R}, d_f)$:

For each compact set K there is a closed ball S and there is a compact set L such that $K \subseteq \mathbb{R} - S \subseteq L$. (1.2)

2. MAIN RESULTS.

THEOREM 1. Let $f \in \mathcal{M}$. Suppose that there are $g, h : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that g, h are nonincreasing, and they are not constant in each neighborhood of the point $+\infty$, (2.1)

$g(x) \leq f(x) \leq h(x)$ in some neighborhood of the point $+\infty$, (2.2)

$\lim_{x \to +\infty} g(x) = \lim_{x \to +\infty} h(x)$. (2.3)

Then the metric space $(\mathbb{R}, d_f)$ has the property (1.2).

PROOF. Let $m \in \mathbb{N}$ be such that $g(x) \leq f(x) \leq h(x)$ for each $x \in [m, \infty)$. Put $d = \lim_{x \to +\infty} g(x)$. Evidently $d = \lim_{x \to +\infty} f(x) > 0$. Let K be a compact set. Put: $s = \inf K - m$, $r = \sup K - s$, $\varepsilon = g(r)$. Since g is not constant on $(r, +\infty)$, there is $\xi > r$ such that $g(\xi) \neq \varepsilon$. Since g is nonincreasing, we have $\varepsilon \neq g(\xi) \leq g(r) = \varepsilon$. Therefore $g(\xi) < \varepsilon$. Since g is nonincreasing for each $x \geq \xi$ we get $g(x) \leq g(\xi)$. Thus $d = \lim_{x \to +\infty} g(x) \leq g(\xi) < \varepsilon$. Let $x \in [m, r]$. Then $f(x) \geq g(x) \geq g(r) = \varepsilon$. Therefore

$$\forall x \in [m, r] : f(x) \geq \varepsilon. \tag{2.4}$$

Let $\delta \in (d, \varepsilon)$. Since $\lim_{x \to +\infty} h(x) = d < \varepsilon$, there is $t > r$ such that $h(t) < \delta$. Let $x \geq t$. Then $f(x) \leq h(x) \leq h(t) < \delta$. Thus

$$\forall x \in [t, \infty) : f(x) < \delta. \tag{2.5}$$

Let S be a closed ball with the centre s and the radius δ. Put $L = [s - t, s + t]$. Now, we shall show that $K \subseteq \mathbb{R} - S$. Let $u \in K$. Then $|u - s| = u - s \in [m, r]$, and by (2.4) we get $d_f(u, s) = f(|u - s|) \geq \varepsilon > \delta$. Therefore $u \notin S$. Finally, we shall show that $\mathbb{R} - S \subseteq L$. Let $v \in \mathbb{R} - S$. Then $f(|v - s|) = d_f(v, s) > \delta$. By (2.5) we have $|v - s| < t$. Therefore $v \in L$.

EXAMPLE 2. Define $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ as follows:

$$f(x) = x, if x \in [0, 1), f(x) = \frac{1 + x + \sin^2(x - 1)}{2x}, \text{ if } x \in [1, \infty).$$

It is not difficult to verify that $f \in \mathcal{M}$ and the metric space $(\mathbb{R}, d_f)$ has the property (1.2) (which yields also the property (1.1)), however f is not monotone on every neighborhood of the point $+\infty$.

EXAMPLE 3. Define $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ as follows; $f(x) = x$, if $x \in [0, 1]$, and $f(x) = \frac{1}{2}(x - 3n + 1 - |x - 3n + 1| + |x - 3n + \frac{1}{2} + \frac{1}{n+2}| + |x - 3n - \frac{1}{2} - \frac{1}{n+2}|)$, if $x \in (3n - 2, 3n + 1]$ $(n = 1, 2, 3, \ldots)$. It is not difficult to verify that $f \in \mathcal{M}$ and $(\mathbb{R}, d_f)$ is a metric space with the property (1.1), which has not the property (1.2). Indeed, the intersection of the sequence of closed balls $\{S_n\}_{n=1}^{\infty}$ (where S_n has the centre $x_n = 3.\ (2^{n-1} - 1)$ and the radius $\varepsilon_n = \frac{1}{2} + \frac{1}{2^n+1}$) is empty.

A characterization of metric preserving functions f such that the space $(\mathbb{R}, d_f)$ has the property (1.1) remains an open question.

ACKNOWLEDGEMENT: The first named author wishes to express his appreciation to the Department of Mathematics, Youngstown State University for their hospitality during his stay. The second named author wishes to acknowledge a support from Youngstown State University Research Council.

REFERENCES

1. BORSÍK, J. and DOBOŠ, J. Functions whose composition with every metric is a metric, Math. Slovaca 31(1981), 3-12 (in Russian).
2. BORSÍK, J. and DOBOŠ, J. On metric preserving functions, Real Analysis Exchange 13 (1987-88), 285-293.
3. TERPE, F. Metric preservice fucntiosn, Proc. Conf. Topology and Measure IV, Greifswald, 1984, 189-197.
4. Juza, M. Poznámka o úplných metrických prostoroch, Matematicko-fyzikálny časopis SAV, 6(1956), 143-148.