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A SURVEY OF METRIC PRESERVING FUNCTIONS

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INTRODUCTION

Let us begin by recalling that a function $f : X \longrightarrow \mathbb{R}^+$ is said to be subadditive if it satisfies the inequality $f(x+y) \leqq f(x) + f(y)$ whenever $x, y \in X$, where X is an additive monoid and $\mathbb{R}^+ = [0, +\infty)$. (See [27] and [34].) It turns out that subadditivity admits a nice characterization in terms of infimal convolution. If $f, g : X \longrightarrow \mathbb{R}^+$, then their infimal convolute $f_\Box g$ is the function defined on X that sends each $x \in X$ to the real number

$$(f_\Box g)(x) = \inf\{f(y) + g(z); y, z \in X \text{ and } y + z = x\}.$$

(See [21] and [30].) In the following theorem, $f \wedge g$ denotes the meet of the functions $f, g : X \longrightarrow \mathbb{R}^+$, $(f \wedge g)(x) = \min\{f(x), g(x)\}$ for each $x \in X$.

Theorem 1. (See [30].) *Let $f, g : X \longrightarrow \mathbb{R}^+$ with $f(0) = 0 = g(0)$. Then the following statements hold :*

(a) *f is subadditive iff $f_\Box f = f$,*
(b) *if $f \wedge g$ is subadditive, then $f_\Box g = f \wedge g$, and*
(c) *if f and g are both subadditive, then $f_\Box g$ is the largest subadditive minorant of $f \wedge g$.*

For recent results on subadditive functions, see [19] and [20]. Let us mention only one of the results proved there :

Every subadditive and right-continuous bijection of $\mathbb{R}^+$ is a homeomorphism.

METRIC PRESERVING FUNCTIONS

Let (M, d) be a metric space. For each $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ define the function $d_f : M \times M \longrightarrow \mathbb{R}^+$ as follows

$$d_f(x, y) = f(d(x, y)) \text{ for each } x, y \in M.$$

We call a function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ metric preserving iff for each metric space (M, d) the function d_f is a metric on M. For example, we can derive a bounded metric from a given metric by the function $x \mapsto \frac{x}{1+x}$.

Denote by $\mathcal{O}$ the set of all functions $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with $f(0) = 0$. The following result is simple and well known. (See [35], and [17], p. 131.)

Proposition 1. *If $f \in \mathcal{O}$ is a nondecreasing, subadditive function vanishing exactly at the origin, then it is metric preserving.*

Simple examples of such functions are concave functions. (See [2], [28], and [29].) Let us recall that a function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is called concave whenever the inequality

$$f\left(\frac{x+y}{2}\right) \geqq \frac{f(x)+f(y)}{2}$$

holds for all $x, y \in \mathbb{R}^+$. It is not difficult to verify that each concave function $f \in \mathcal{O}$ is nondecreasing and continuous on $(0, +\infty)$. (See [14].) It is easy to see that for each $f \in \mathcal{O}$ the following statements hold :

(i) if f is concave, then $x \mapsto \frac{f(x)}{x}$ is nonincreasing on $(0, +\infty)$, and

(ii) if $x \mapsto \frac{f(x)}{x}$ is nonincreasing on $(0, +\infty)$, then f is subadditive.

Therefore each concave function $f \in \mathcal{O}$ is metric preserving.

Let us recall that a modulus of continuity is a function $f \in \mathcal{O}$ such that f is nondecreasing, subadditive, and continuous from the right at the origin (compare to Proposition 1). In [10] it is shown that the standard Cantor function ϕ ("the devil's staircase") is a modulus of continuity. It is not difficult to verify that $x \mapsto \frac{\phi(x)}{x}$ is not nonincreasing on $(0, +\infty)$.

Let a,b and c be positive real numbers. We call the triplet (a, b, c) a triangle triplet iff $a \leqq b+c$, $b \leqq a+c$, and $c \leqq a+b$. (See [31].) The following theorem gives a characterization of metric preserving functions, which is based on the fact that each three-points space has a representation by certain subspace of the Euclidean plane. (See [2], [8], [29], and [31].)

Theorem 2. *Let $f \in \mathcal{O}$. Then f is metric preserving iff f vanishes exactly at the origin, and it has the following property :*

if (a, b, c) is a triangle triplet, then so is $(f(a), f(b), f(c))$.

Corollary. *Every metric preserving function is subadditive.*

It is well known that there is a complete metric space with the following property

(A) *There is a monotone sequence of closed balls the intersection of where is empty.*

In the paper [16] such metric space has been constructed by a modification of the Euclidean metric on the real line.

Example 1. Define $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ as follows

$$f(x) = \begin{cases} x & x \leqq 2, \\ 1 + \frac{1}{x-1} & x > 2. \end{cases}$$

In [16] it is shown that f is metric preserving and the metric space $(\mathbb{R}, e_f)$ has the property (A), where e is the Euclidean metric on $\mathbb{R}$ (i.e. $e(x, y) = |x - y|$ for each $x, y \in \mathbb{R}$). The proof of (A) is based on the following property of the metric space $(\mathbb{R}, e_f)$:

(B) *For each compact set K there is a closed ball S and there is a compact set L such that $K \subset \mathbb{R} - S \subset L$.*

In [12] it is shown that there is a metric preserving function f such that the metric space $(\mathbb{R}, e_f)$ has the property (A), but not the property (B).

A characterization of metric preserving functions f such that the metric space $(\mathbb{R}, e_f)$ has the property (A) is an open question.

The following result shows an importance of continuity in our investigations.

Theorem 3. (See [2].) *Let (M, d) be a metric space. Let f be a metric preserving function. If f is continuous, then the metrics d and d_f are uniformly equivalent. If (M, d) is not discrete (is not uniformly discrete), then the metrics d and d_f are equivalent (uniformly equivalent) iff f is continuous.*

The following theorem gives a characterization of the continuous metric preserving functions.

Theorem 4. (See [2].) *Let f be metric preserving function. Then the following three conditions are equivalent :*

(1) *f is continuous,*
(2) *f is continuous on the right at the origin,*
(3) $\forall \varepsilon > 0 \exists x > 0 : f(x) < \varepsilon$.

Differentiable properties of metric preserving functions are studied in [5], [11], and [31]. The following theorem is the main result of the paper [5] (compare to [35] for monotone functions).

Theorem 5. *Let f be a metric preserving function. Then $f'_+(0)$ exists (finite or infinite), and*

$$f'_+(0) = \inf\{k > 0 : f(x) \leqq k.x \text{ for each } x \in \mathbb{R}^+\}.$$

If $f'_+(0) < +\infty$, then f is a Lipschitz mapping with the Lipschitz constant $f'_+(0)$ (which yields that f is differentiable almost everywhere).

In contrast with this property it is shown in [11], that there is a continuous, metric preserving function which is nowhere differentiable. This function is a slight modification of the Van der Waerden's continuous nowhere differentiable function.

Metric preserving functions f such that $f - id$ is periodic, where id is the identity function on $\mathbb{R}^+$ (i.e. $id(x) = x$ for each $x \in \mathbb{R}^+$), are studied in [25]. Some other properties of metric preserving functions are contained in [6], [7], [13], [24], and [33].

Metric preserving functions of several variables

Let (M_1, d_1) and (M_2, d_2) be two metric spaces. It is well known that $\max(d_1, d_2)$, $\sqrt{d_1^2 + d_2^2}$, and $d_1 + d_2$ are metrics on $M_1 \times M_2$. In these cases we obtain new metrics as composite functions of the real functions $(x, y) \mapsto \max(x, y)$, $(x, y) \mapsto \sqrt{x^2 + y^2}$, and $(x, y) \mapsto x + y$, respectively, with the "vector-metric" $d : (M_1 \times M_2)^2 \longrightarrow (\mathbb{R}^+)^2$, where $d((p, q), (r, s)) = (d_1(p, r), d_2(q, s))$.

We shall generalize this idea.

Let T be a nonempty set of indices. Consider the indexed family $\{(M_t, d_t)\}_{t\in T}$ of metric spaces. Define $d : (\prod_{t\in T} M_t)^2 \longrightarrow (\mathbb{R}^+)^T$ as follows

$$d((x_t)_{t\in T}, (y_t)_{t\in T}) = (d_t(x_t, y_t))_{t\in T}.$$

We say that the function $f : (\mathbb{R}^+)^T \longrightarrow \mathbb{R}^+$ is a metric preserving function if for each indexed family $\{(M_t, d_t)\}_{t\in T}$ of metric spaces the composite function $f(d)$ is a metric on the set $\prod_{t\in T} M_t$.

Let us recall that a function $f : (\mathbb{R}^+)^T \longrightarrow \mathbb{R}^+$ is called isotone iff $f((x_t)_{t\in T}) \leqq f((y_t)_{t\in T})$ whenever $0 \leqq x_t \leqq y_t$ for each $t \in T$.

The following sufficient condition is a generalization of Proposition 1.

Proposition 2. (See [22].) *If $f : (\mathbb{R}^+)^T \longrightarrow \mathbb{R}^+$ is an isotone, subadditive function vanishing exactly at the constant zero function, then it is metric preserving.*

The following theorem gives a characterization of metric preserving functions. (See [3].)

Theorem 6. *Let $f : (\mathbb{R}^+)^T \longrightarrow \mathbb{R}^+$. Then f is metric preserving iff it is a function vanishing exactly at the constant zero function and it has the following property :*

if (a_t, b_t, c_t) are triangle triplets for each $t \in T$, then $(f((a_t)_{t\in T}), f((b_t)_{t\in T}), f((c_t)_{t\in T}))$ is a triangle triplet.

Consider the indexed family $\{(M_t, d_t)\}_{t\in T}$ of metric spaces. Denote by $\mathcal{T}$ the product topology on $\prod_{t\in T} M_t$. Let $f : (\mathbb{R}^+)^T \longrightarrow \mathbb{R}^+$ be a metric preserving function. Denote by $\mathcal{T}_f$ the topology on $\prod_{t\in T} M_t$ generated by the metric $f(d)$. A natural question arises whether we can investigate metrizability of the product topology by the metric $f(d)$.

Theorem 7. (See [3].) *$\mathcal{T} = \mathcal{T}_f$ iff*

$$\begin{gathered}\forall \varepsilon > 0\ \exists \delta > 0\ \forall (n_t)_{t\in T} \in \mathbb{N}^T\ \exists (a_t)_{t\in T} \in (\mathbb{R}^+)^T : \\ \forall t \in T - (I \cup F) : a_t \geqq n_t, \\ \forall t \in I - F : a_t \geqq \operatorname{diam} M_t, \\ \forall t \in F \cap H : a_t \geqq \delta, \\ f((a_t)_{t\in T}) < \varepsilon,\end{gathered}$$

where $I = \{t \in T;$ the metric space (M_t, d_t) is bounded $\}$, and $H = \{t \in T;$ the metric space (M_t, d_t) is not discrete $\}$.

Denote by $\mathcal{U}$ the product uniformity on $\prod_{t\in T} M_t$. Denote by $\mathcal{U}_f$ the uniformity on $\prod_{t\in T} M_t$ generated by the metric $f(d)$. By a similar way one can derived a characterization of metric preserving functions f with $\mathcal{U} = \mathcal{U}_f$. (See [4].) If f is continuous, then $\mathcal{U} = \mathcal{U}_f$ (which yields $\mathcal{T} = \mathcal{T}_f$). Some other properties of metric preserving functions of several variables are contained in [32].

In the rest of this paper we will suppose that the set of indices is finite. Let $f : (\mathbb{R}^+)^n \longrightarrow \mathbb{R}^+$. Define the functions $f_i : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ $(i = 1, \ldots, n)$ as follows

$$f_1(x) = f(x, 0, \ldots, 0),$$
$$f_2(x) = f(0, x, 0, \ldots, 0),$$
$$\vdots$$
$$f_n(x) = f(0, \ldots, 0, x)$$

Proposition 3. (See [9].) *Let $f : (\mathbb{R}^+)^n \longrightarrow \mathbb{R}^+$ be metric preserving. Then f is continuous iff all f_i $(i = 1, \ldots, n)$ are continuous.*

The following result is obtained by [15] and [18].

Theorem 8. *Let $\|.\|$ be a norm in $\mathbb{R}^n$. Then the restriction of $\|.\|$ to $(\mathbb{R}^+)^n$ is metric preserving iff it is isotone.*

Some applications of such functions concerning intrinsic metrics are contained in [15] and [23]. Isotone metric preserving functions $f : (\mathbb{R}^+)^n \longrightarrow \mathbb{R}^+$ such that $f_i = id$ for each $i \in \{1, \ldots, n\}$ are studied in [1], pp. 196-198, and [26].

Let $(M_1, d_1), \ldots, (M_n, d_n)$ be metric spaces. Let us recall that $H = \{i;\ (M_i, d_i)$ is not discrete $\}$.

Theorem 9. (See [9].) *Let $f, g : (\mathbb{R}^+)^n \longrightarrow \mathbb{R}^+$ be metric preserving. Then $\mathcal{T}_f \subset \mathcal{T}_g$ iff for each $i \in H$ the following statement holds :*
if g_i is continuous, then f_i is continuous.

Put $\mathcal{L} = \{\mathcal{T}_f;\ f : (\mathbb{R}^+)^n \longrightarrow \mathbb{R}^+$ is a metric preserving function $\}$.

Theorem 10. (See [9].) *The lattice $\mathcal{L}$ is dually isomorphic to the lattice* $(\exp H, \subset)$.

A characterization of the lattice of topologies $\mathcal{T}_f$ on an infinite products of metric spaces is an open question.

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