PREIMAGES OF BAIRE SPACES

JOZEF DOBOŠ, Košice, Z. PIOTROWSKI, Youngstown, I. L. REILLY, Auckland

(Received May 14, 1993)

Summary. A simple machinery is developed for the preservation of Baire spaces under preimages. Subsequently, some properties of maps which preserve nowhere dense sets are given.

Keywords: Baire spaces, functions preserving nowhere dense sets

AMS classification: 54C10 (54C05)

I. INTRODUCTION

Much is known about the invariance of Baire spaces under mappings. In particular, the images of Baire spaces under functions which are open and continuous, are Baire [Du], see also [Do], [Fr], [Ne], and [PS] for more refined results related to some weaker forms of the openness and/or continuity of functions.

Not so well-known are a few, sparse results on the preservation of Baire spaces under preimages, see [Fr], [No] and [HM].

We shall first recall a famous problem of R. Sikorski, namely whether a product of two metric Baire spaces is Baire.

This difficult problem of almost 40 years standing has been solved in the negative [FK]. More specifically, a metric Baire space X has been constructed such that its square X^2 is of the first category.

The following Lemma is an immediate corollary of the above result. Recall that a function $f\colon X \to Y$ is called Baire-fiber if $f^{-1}(y)$ is a Baire subspace of X for each point y in Y.

Lemma 0. *There is a first category metric space X, a metric Baire space Y and an open, continuous and Baire-fiber function from X onto Y.*

Proof. Let f be the projection $pr\colon X^2 \to X$ where X is the Fleissner-Kunen example mentioned above. As a projection it is continuous and open. For each x in X, $f^{-1}(x)$ is Baire, since it is the subspace $\{x\} \times X$, which is homeomorphic to X. □

So, if even open and continuous functions having Baire fiber do not preserve Baireness under preimages even in the setting of metric spaces, what can we expect?

II. Functions preserving nowhere dense sets

The following remarkably simple technique, seems to have been overlooked in the past. Let us start from

Definition. Given a space X, let $f\colon X \to f(X)$ be a function. We say that f is *nowhere dense sets preserving* (abbreviated henceforth ad *nd*-preserving) if the image of a nowhere dense set in X is a nowhere dense set in $f(X)$.

Functions which are nd-preserving have been studied in [Wi] under the name B1 functions, in connections with (strong) transitivity.

Lemma 1. *Functions which are nd-preserving send sets of the first category onto sets of the first category.*

Proof. Let $f\colon X \to Y$ be an nd-preserving function. Suppose F is of the first category in X, i.e., $F = \bigcup_{i=1}^{\infty} N_i$, where N_i are nowhere dense, $i = 1, 2, \ldots$.

Now,

$$f(F) = f\left(\bigcup_{i=1}^{\infty} N_i\right) = \bigcup_{i=1}^{\infty} f(N_i).$$

□

Corollary 1. *nd-preserving functions preserve spaces of the first category under images.*

The following corollary follows from a recently published result of Fitzpatrick and Zhou [FZ], Lemma 3.1.

Corollary 2. *If X is a homogeneous, non-Baire space and $f\colon X \to Y$ is nd-preserving, then Y is of the first category.*

Proof. Such X is then of the first category. Now, apply Corollary 1. □

Corollary 3. *nd-preserving functions preserve spaces of the second category under preimages.*

Proof. Let $f: X \to Y$ be a function. If $f(X)$ is not of the first category (that is, of the second category), then X is not of the first category, i.e., of the second category. □

Corollary 4. *If X is a homogeneous space, Y is of the second category, and if $f: X \to Y$ is nd-preserving, then X is Baire.*

Proof. X is then of the second category and being homogeneous is Baire. □

Corollary 5. *Let X be a topological group, let Y be of the second category, and let $f: X \to Y$ be nd-preserving. Then X is Baire.*

So, do nd-preserving functions preserve Baireness under preimages?

No! And here is a counterexample. This example also shows the *necessity of homogeneity* of X in Corollary 4 (hence also in Corollary 5).

Example 1. Let X be the disjoint topological sum of $\{(x,y) \in \mathbb{R}^2 : y \neq 0\}$ and $\{(x,0) \in \mathbb{R}^2 : x \in \mathbb{Q}\}$, and let $Y = \{(x,y) \in \mathbb{R}^2 : y \neq 0\} \cup \{(x,0) \in \mathbb{R}^2 : x \in \mathbb{Q}\}$, which the relative topology from $\mathbb{R}^2$. Clearly, Y is a Baire space whereas X is not. Now, the identity function f from X to Y is an nd-preserving function.

Although nd-preserving functions do not necessarily preserve Baire spaces under inverse images, nonetheless they do so in the presence of feeble openness.

Recall that a function $f: X \to f(X)$ is called *feebly open* (resp. *feebly continuous*) if the image (resp. inverse image) under f of any nonempty open set has a nonempty interior.

Proposition 1. *If f is a feebly open nd-preserving function from a space X onto a Baire space Y, then X is a Baire space.*

Proof. Suppose that V is an open, first category subset in S. Since f is nd-preserving, $f(V)$ is of the first category in Y, by Lemma 1. Now, since f is feebly open, there is a set U, open in Y, such that $U \subset f(V)$. Thus U is an open first category subset in Y, which is a contradiction. □

How does the nd-preserving property relate to continuity?

Are all continuous functions nd-preserving?

No! We provide two different examples of continuous functions which are not nd-preserving.

Example 2. Let f be the projection $pr\colon X^2 \to X$ considered in the proof of Lemma 0. It is continuous. Being open f is obviously feebly open. So we can apply Proposition 1. Since the range X is Baire, if f is nd-preserving then X^2 would be Baire, which is a contradiction.

Example 3. Let the topology $\mathscr{T}$ on the set of reals $\mathbb{R}$ be generated by $\mathscr{E} \cup \{U \cap \mathbb{Q}\colon U \in \mathscr{E}\}$, where $\mathscr{E}$ is the Euclidean topology. Now, $(\mathbb{R}, \mathscr{T})$ is not a Baire space, $(\mathbb{R}, \mathscr{E})$ is Baire and the identity function $i\colon (\mathbb{R}, \mathscr{T}) \to (\mathbb{R}, \mathscr{E})$ is continuous, since $(\mathbb{R}, \mathscr{T})$ has more open sets. Now, i is not nd-preserving, let us observe that $\mathrm{cl}_{\mathscr{T}}\big[(\mathbb{R} \setminus \mathbb{Q}) \cap [0,1]\big] = (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1]$ shows that the latter set is nowhere dense in $\mathscr{T}$-topology, whereas $\mathrm{int}_{\mathscr{E}}\, \mathrm{cl}_{\mathscr{E}}\big[(\mathbb{R} \setminus \mathbb{Q}) \cap [0,1]\big] = (0,1)$ which is not nowhere dense in the Euclidean topology.

Are all nd-preserving functions continuous?

Again, no!

Example 4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x - [x]$ for each $x \in \mathbb{R}$, where $[x]$ denotes the greatest integer less than or equal to x. Clearly, f has infinitely many points of discontinuity and sends nowhere dense sets onto nowhere dense sets.

III. Real-valued nd-preserving functions of real variable

The symbols $L^-(f,a)$, $L^+(f,a)$ denote the cluster sets from the left and right, respectively, of the function $f\colon \mathbb{R} \to \mathbb{R}$ at the point a i.e., $r \in L^-(f,a) \Leftrightarrow \exists\{x_n\}\colon x_n \to a,\ x_n < a,\ f(x_n) \to r$.

Lemma 2. *Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ has the Darboux property. Let f be nd-preserving.*

If $a \in (-\infty, +\infty]$, then the set $L^-(f,a)$ is a singleton.

If $a \in [-\infty, +\infty)$, then the set $L^+(f,a)$ is a singleton.

Proof. Suppose to the contrary $r,\ s \in L^-(f,a)$, $r < s$. Then there are sequences $\{x_n\}$, $\{y_n\}$ such that

$$x_n \to a, \quad x_n < a, \quad f(x_n) \to r,$$
$$y_n \to a, \quad y_n < a, \quad f(y_n) \to s.$$

Let $p,\ q \in \mathbb{R}$ be such that $r < p < q < s$. Let $V = \{v_1, v_2, v_3, \ldots\}$ be a countable dense subset of the interval (p,q). We will show that there is a discrete set $U \subset \mathbb{R}$ such that $f(U) = V$.

Let $k \in \mathbb{N}$ be such that $x_k > a - 1$, $f(x_k) < p$. Let $m \in \mathbb{N}$ be such that $y_m > x_k$, $f(y_m) > q$. It follows from the Darboux property of f on (x_k, y_m) that there is $u_1 \in (x_k, y_m)$ such that $f(u_1) = v_1$.

By induction, we may construct a sequence $\{u_n\}$ with

$$a - \frac{1}{n} < u_n < a, \quad n \in \mathbb{N},$$

$$f(u_n) = v_n, \quad n \in \mathbb{N}.$$

Put $U = \{u_n : n \in \mathbb{N}\}$. Then U is a discrete subset of $\mathbb{R}$ with $f(U) = V$. □

The second part of the proof is similar.

The following Proposition 2 follows directly from Lemma 2.

Proposition 2. *If $f: \mathbb{R} \to \mathbb{R}$ is an nd-preserving function with the Darboux property, then f is continuous.*

Corollary 6. *Let $f: \mathbb{R} \to \mathbb{R}$ be an nd-preserving function with the Darboux property. Then the set $L^-(f, +\infty)$ is a singleton and the set $L^+(f, -\infty)$ is a singleton.*

Example 5. Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \sin x$, $x \in \mathbb{R}$. Then f is not nd-preserving, even though it is continuous.

The following example shows that there is a continuous *monotone* function $f: [0, 1] \to [0, 1]$ which is not nd-preserving.

Example 6. Let $f: [0, 1] \to [0, 1]$ be the standard Cantor function. Then f is a continuous nondecreasing map of $[0, 1]$ onto $[0, 1]$. Denote by C the standard Cantor set. Then C is nowhere dense in $[0, 1]$, but $f(C)$ is dense in $[0, 1]$.

IV. Feeble homeomorphisms and nd-preserving functions

In view of Proposition 1, nd-preserving functions play an important role in preservation of Baire Category under preimages. However, as the above examples show the nd-preserving property does not come easy, see Lemma 0, and compare it with Proposition 1.

Nevertheless, the following is true:

Proposition 3. *Let $f: X \to Y$ be a feeble homeomorphism. Then f is nd-preserving.*

Proof. Let N be a nowhere dense subset of X. Let U be a nonempty open subset of Y. By feeble continuity of f, $\operatorname{int} f^{-1}(U) \neq \emptyset$. Since N is nowhere dense, there is a set W, open in X, such that

(*) $W \cap N = \emptyset$, and

(**) $W \subset \operatorname{int} f^{-1}(U)$.

Since f is feebly open, $\operatorname{int} f(W) \neq \emptyset$. Obviously, $\operatorname{int} f(W) \cap f(N) \subset f(W) \cap f(N)$. By the injectivity of f and (*) we get

$$f(W) \cap f(N) = f(W \cap N) = \emptyset.$$

So, $\operatorname{int} f(W) \cap f(N) = \emptyset$.

Now, (**) implies

$$f(W) \subset f\big(\operatorname{int} f^{-1}(U)\big) \subset f\big(f^{-1}(U)\big) = U.$$

Thus $\operatorname{int} f(W) \subset U$, since U is open in Y.

Hence, for any nonempty open set U, we have a nonempty open subset $\big(\operatorname{int} f(W)\big)$ that misses $f(N)$, which shows that $f(N)$ is nowhere dense in Y. □

Corollary 7 [Do]. *If f is a feeble homeomorphism from a space X onto a Baire space Y, then X is a Baire space.*

Proof. Such an f is nd-preserving. Now apply Proposition 1. □

So, bijectivity of a function in the presence of feeble openness and feeble continuity enforces the nd-preserving property. If the inverses of points are points then a feebly open and feebly continuous function is nd-preserving.

Now, let us assume that inverse images of points are, say, compact ("compact sets behave like points!").

This has been done in [No]. But the proof *does not refer* to the nd-preserving property. Rather it uses game-theoretic properties of Baire Category.

Similarly, Frolík [Fr] obtained his result by imposing some restrictions upon X, in terms of the countability of its base, and then proved his result, on a piecemeal basis without referring to the nd-preserving property.

In view of the above remarks, it is interesting to see whether both the fiber-complete case of D. Noll [No] and the case restrictions case of Frolík [Fr] can be derived as corollaries to Proposition 1, i.e., if the functions considered turn out to be nd-preserving. Or, whether counter examples can be provided to this hypothesis.

References

[Do] *J. Doboš*: A note on the invariance of Baire spaces under mappings. Časopis pěst. matem. *108* (1983), 409–411.

[Du] *J. Dugundji*: Topology. Wm. C. Brown Publishers, 1989, pp. 256.

[FZ] *B. Fitzpatrick, Jr. and Zhou Hao-xuan*: Countable dense homogeneity and the Baire property. Topology Appl. *43* (1992), 1–4.

[FK] *W. G. Fleissner and K. Kunen*: Barely Baire spaces. Fund. Math. *101* (1979), 229–240.

[Fr] *Z. Frolík*: Remarks concerning the invariance of Baire spaces under mappings. Czechoslovak Math. J. *11* (1961), 381–385.

[HM] *R. C. Haworth and R. A. McCoy*: Baire spaces. Dissertationes Math., PWN, Warszawa *141* (1977).

[Ne] *T. Neubrunn*: A note on mappings of Baire spaces. Math. Slovaca *27* (1977), 173–176. Erratum, ibidem, p. 442.

[No] *D. Noll*: On the preservation of Baire category under preimages. Proc. Amer. Math. Soc. *107* (1989), 847–854.

[PR] *Z. Piotrowski and I. L. Reilly*: Preimages of Baire spaces – an example. Q & A in General Topology *11* (1993), 105–107.

[PS] *Z. Piotrowski and A. Szymański*: Some remarks on category in topological spaces. Proc. Amer. Math. Soc. *101* (1987), 156–160.

[Wi] *T. E. Wilson*: Strictly irreducible maps and strong transitivity. Topology Proceedings *13* (1988), 371–387.

Authors' addresses: *Jozef Doboš*, Technická Univerzita, Katedra matamatiky SjF, Letná 9, 040 01 Košice, Slovak Republic; *Z. Piotrowski*, Department of Mathematics, Youngstown State University, Youngstown, OH 44555 USA; *I. L. Reilly*, Department of Mathematics and Statistics, University of Auckland, Auckland, New Zealand.