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Ján Borsík, Matematyký ústav SAV, Grešákova 6, 040 01 Košice, Czechoslovakia

Jozef Doboš, Katedra matematiky SjF VŠT, Švermova 9, 040 01 Košice, Czechoslovakia

ON SIMPLE CONTINUITY POINTS

Throughout this paper we assume that X and Y are topological spaces. The letters $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ stand for the set of natural, rational and real numbers, respectively.

N. Biswas in [1] introduced the following concept of simple continuity.

Definition 1. *A function $f : X \to Y$ is said to be simply continuous if for every open set V in Y the set $f^{-1}(V)$ is a union of an open set in X and a nowhere dense set in X.*

The purpose of the present paper is to introduce a suitable pointwise definition of that notion and to give a characterization of the set of all simple continuity points.

Definition 2. *We say that $f : X \to Y$ is simply continuous at a point $x \in X$ if for each open neighborhood V of $f(x)$ and for each neighborhood U of x the set $f^{-1}(V) \setminus \mathrm{int}\, f^{-1}(V)$ is not dense in U. Denote by N_f the set of all points at which f is simply continuous.*

REMARK 1. Let $f : X \to Y$. It is easy to verify that

(α) f is simply continuous in the sense of Biswas if and only if $N_f = X$,

(β) $Q_f \subset N_f$, where Q_f denotes the set of all points at which f is quasicontinuous (see [8]).

Lemma 1. *Let $f : X \to Y$. Then for each open set V in Y the set $N_f \cap (f^{-1}(V) \setminus \mathrm{int}\, f^{-1}(V))$ is nowhere dense in X.*

PROOF. Let V be an open set in Y. Put $W = f^{-1}(V) \setminus \mathrm{int}\, f^{-1}(V)$. It is easy to see that $W \cap \mathrm{int}\,\mathrm{cl}\, W \subset X - N_f$. Hence the set $N_f \cap W \subset (N_f \cap W) \setminus \mathrm{int}\,\mathrm{cl}\, W \subset W \setminus \mathrm{int}\,\mathrm{cl}\, W$ is nowhere dense in X.

Proposition 1. *Let $f : X \to Y$, where Y is second countable. Then the set $N_f \setminus C_f$ (where C_f is the set of all continuity points of f) is of the first category in X.*

PROOF. Let $\{B_n : n \in \mathbb{N}\}$ be a countable base of open sets in Y. Since $X \setminus C_f = \bigcup_{n=1}^{\infty} (f^{-1}(B_n) = \text{ int } f^{-1}(B_n))$, by Lemma 1 the set $N_f \setminus C_f = \bigcup_{n=1}^{\infty} (N_f \cap (f^{-1}(B_n) \setminus \text{ int } f^{-1}(B_n)))$ is of the first category in X.

The following example shows that the set $N_f \setminus C_f$ may be dense in the domain of f.

EXAMPLE 1. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = r(x) + x$, where $r : \mathbb{R} \to \mathbb{R}$ is the Riemann function defined by

$$r(x) = \begin{cases} \frac{1}{q}, & \text{for } x = \frac{p}{q} \text{ (where } p, q \text{ are relatively prime, } q > 0), \\ 0 & \text{otherwise.} \end{cases}$$

Then $N_f \setminus C_f = \mathbb{Q}$ is dense in $\mathbb{R}$.

Definition 3. (See [8]). *Let $f : X \to Y$, where Y is a metric space with a metric d. We say that f is cliquish at a point $x \in X$ if for each $\varepsilon > 0$ and each neighborhood U of x there is a nonempty open set $G \subset U$ such that $d(f(x), f(y)) < \varepsilon$ for each $y, z \in G$. Denote by A_f the set of all points at which f is cliquish. If $A_f = X$, then f is said to be cliquish.*

REMARK 2. Let $f : X \to Y$, where Y is a metric space. Then the set $A_f \setminus N_f \subset A_f \setminus C_f$ is of the first category (see [10]). If Y is separable, then according to Proposition 1 the set $N_f \setminus A_f$ is of the first category.

The following example shows that the set $N_f \setminus A_f$ may be uncountable.

EXAMPLE 2. Let C be the Cantor discontinuum. Let $\chi : \mathbb{R} \to \mathbb{R}$ be the Dirichlet functions (i.e. $\chi(x) = 1$ for $x \in \mathbb{Q}$ and $\chi(x) = 0$ otherwise). Define $f : \mathbb{R} \to \mathbb{R}$ on the contiguous intervals (a, b) of C as follows

$$f(x) = \begin{cases} 1 + \chi(x), & \text{for } x \in (a, a + \frac{1}{3}(b-a)), \\ 2\chi(x), & \text{for } x \in (a + \frac{1}{3}(b-a), a + \frac{2}{3}(b-a)), \\ \chi(x), & \text{for } x \in (a + \frac{2}{3}(b-a), b), \end{cases}$$

and $f(x) = 0$ otherwise.

Then $N_f \setminus A_f = C \setminus \{0, 1\}$ is uncountable.

Theorem 1. Let $f : X \to Y$, where Y is a metric space with a metric d. Let at least one of the following conditions be satisfied:

(i) X is a Baire space and Y is a separable metric space,

(ii) Y is a totally bounded metric space.

Then the set $N_f \setminus A_f$ is nowhere dense in X.

PROOF. Put $G =$ int cl$(N_f \setminus A_f)$. We shall show that $G = \phi$. Suppose, by way of contradiction, that $G \neq \phi$. Put $K = G \setminus A_f$. Since the set A_f is closed (see[7]) the set K is open. We shall show that $K \neq \phi$. Since the set int $A_f \cup (X \setminus A_f)$ is dense in X and $G \cap \text{ int } A_f = \text{int}(\text{cl}(N_f \setminus A_f) \cap A_f) \subset \text{int}(A_f \setminus \text{int } A_f) = \phi$, we get $\phi \neq G \cap (\text{int } A_f \cup (X \setminus A_f)) = (G \cap \text{ int } A_f) \cup K = K$.

Let $x_0 \in K$ be arbitrary. Since $x_0 \notin A_f$, there is $\varepsilon > 0$ and $L \subset K$, an open neighborhood of x_0, such that

(*) for every nonempty open set $M \subset L$ there are $y, z \in M$ such that $d(f(y), f(z)) \geq 8\varepsilon$.

We shall show that there is $v \in Y$ such that $f^{-1}(S(v, \varepsilon))$ is not nowhere dense in L (where $S(a, \eta) = \{t \in Y : d(a, t) < \eta\}$). We distinguish two cases.

a) Suppose that X is a Baire space and Y is separable. Then $Y = \bigcup_{n=1}^{\infty} S(v_n, \varepsilon)$, where $\{v_n : n \in \mathbb{N}\}$ is countable dense set in Y. Since $L = L \cap f^{-1}(\bigcup_{n=1}^{\infty} S(v_n, \varepsilon)) = \bigcup_{n=1}^{\infty} (L \cap f^{-1}(S(v_n, \varepsilon)))$, there is $k \in \mathbb{N}$ such that $L \cap f^{-1}(S(v_k, \varepsilon))$ is not nowhere dense in L.

b) Suppose that Y is totally bounded. Then there is a finite set $\{v_1, v_2, \ldots, v_m\}$ in Y such that $Y = \bigcup_{n=1}^{m} S(v_n, \varepsilon)$. Since $L = L \cap f^{-1}(\bigcup_{n=1}^{m} S(v_n, \varepsilon)) = \bigcup_{n=1}^{m} (L \cap f^{-1}(S(v_n, \varepsilon)))$, there is $k \in \mathbb{N}$ such that $L \cap f^{-1}(S(v_k, \varepsilon))$ is not nowhere dense in L.

Therefore there is a nonempty open set $J \subset L$ such that $f^{-1}(S(v, \varepsilon))$ is dense in J. Put

$$D = \{y \in J : d(f(y), v) \geq 4\varepsilon\}.$$

Then in view of (*) the set D is dense in J. In the following we distinguish two cases.

α) Suppose that there is $x \in J \cap N_f$ such that $d(v, f(x)) > \varepsilon$. Put $B = \{u \in Y : d(u, v) > \varepsilon\}$. Then B is an open neighborhood of $f(x)$. Since

$f(D) \subset B$, the set $f^{-1}(B)$ is dense in J. Since $f^{-1}(S(v,\varepsilon))$ is dense in J and $f^{-1}(S(v,\varepsilon)) \cap f^{-1}(B) = \phi$, we have int $f^{-1}(B) \cap J = \phi$. Therefore $f^{-1}(B) -$ int $f^{-1}(B)$ is dense in J, which contradicts $x \in N_f$.

β) Suppose that $d(v, f(x)) \leq \varepsilon$ for each $x \in J \cap N_f$. Since N_f is dense in J, there is $z \in J \cap N_f$. Then J is an open neighborhood of z and $S(v, 2\varepsilon)$ is an open neighborhood of $f(z)$. Put $V = \{u \in Y : d(u,v) > 2\varepsilon\}$. Since $f(D) \subset V$, the set $f^{-1}(V)$ is dense in J. Since $f^{-1}(S(v,2\varepsilon))$ is dense in J and $f^{-1}(S(v,2\varepsilon)) \cap f^{-1}(V) = \phi$, we have int $f^{-1}(S(v,2\varepsilon)) \cap J = \phi$. Thus $f^{-1}(S(v,2\varepsilon)) \setminus$ int $f^{-1}(S(v,2\varepsilon))$ is dense in J, which contradicts $z \in N_f$.

REMARK 3. Under the assumptions of Theorem 1 every simply continuous function $f : X \to Y$ is cliquish (see [9]). Example 1 in [3] shows that those assumptions cannot be omitted.

Proposition 2. *Under the assumptions of Theorem 1 the set cl $N_f - N_f$ is of the first category in X.*

PROOF. According to Theorem 1, Remark 2 and the fact that A_f is closed (see [7]), the set cl $N_f \setminus N_f \subset$ cl$((N_f \setminus A_f) \cup A_f) \setminus N_f \subset$ cl$(N_f \setminus A_f) \cup (A_f \setminus N_f)$ is of the first category in X.

The following example shows that the assumption "Y is a metric space" in Proposition 2 cannot be omitted.

EXAMPLE 3. Let $Y = \mathbf{R}$, $\mathcal{T} = \{A \subset \mathbf{R} : \mathbf{R} \setminus A$ is finite or $0 \notin A\}$. Then Y is T_4-space. Define $f : \mathbf{R} \to Y$ as follows

$$f(x) = \begin{cases} 0, & \text{for } x \in \mathbf{Q}, \\ x, & \text{otherwise.} \end{cases}$$

Then the set cl $N_f \setminus N_f$ is of the second category in $\mathbf{R}$.

We recall that a subset A of X is almost closed (see [6]) if cl int $A \subset A$.

Proposition 3. *Let $f : X \to Y$. Then the set N_f is almost closed.*

PROOF. Let $x \in$ cl int N_f. Let U be an open neighborhood of x and V an open neighborhood of $f(x)$. We shall show that $f^{-1}(V) -$ int $f^{-1}(V)$ is not dense in U, which yields $x \in N_f$. We distinguish two cases.

a) Suppose that there is $y \in N_f \cap U \cap f^{-1}(V)$. Since $y \in N_f$, the set $f^{-1}(V) \setminus$ int $f^{-1}(V)$ is not dense in U.

b) Suppose that $f^{-1}(V) \cap U \cap N_f = \phi$. Since $x \in \mathrm{cl\,int}\, N_f$, the set $G = U \cap \mathrm{int}\, N_f$ is nonempty open, $G \subset U$ and $f^{-1}(V) \cap G \subset f^{-1}(V) \cap U \cap N_f = \phi$. Therefore $f^{-1}(V) \setminus \mathrm{int}\, f^{-1}(V)$ is not dense in U.

We recall that a topological space X is perfectly normal (see [4], p. 68) if it is normal and each closed subset of X is G_δ. A topological space is resolvable (see [2]) if it is a union of two disjoint dense sets.

Theorem 2. *Let X be a perfectly normal space such that X^d is a resolvable space (where Z^d is the set of all accumulation points of Z). Let Y be a first countable T_1-space such that $Y^d \neq \phi$. Suppose $A \subset X$ is such that*

(1) A contains all isolated points of X,

(2) A is almost closed,

(3) $\mathrm{cl}\, A \setminus A$ is of the first category in X.

Then there is a function $f : X \to Y$ such that $N_f = A$.

PROOF. Let $y_0 \in Y^d$. Let $\{y_n : n \in \mathbb{N}\}$ be a one-to-one sequence which converges to $y_0, y_n \neq y_0$ for all $n \in \mathbb{N}$. Since X^d is resolvable, we can write $X \setminus \mathrm{cl}\, A = B \cup D$, where B and D are disjoint dense sets in $X \setminus \mathrm{cl}\, A$. Since X is perfectly normal, there is a decreasing sequence $\{H_n : n \in \mathbb{N}\}$ of open sets such that $\mathrm{cl}\, A = \bigcap_{n=1}^{\infty} H_n$ and $\mathrm{cl}\, H_{n+1} \subset H_n$ for each $n \in N$. Put $G_0 = \phi$ and $G_n = X \setminus \mathrm{cl}\, H_n$ for each $n \in \mathbb{N}$. Let $\mathrm{cl}\, A \setminus A = \bigcup_{n=1}^{\infty} A_n$, where A_n are mutually disjoint and nowhere dense in X. Define a function $f : X \to Y$ as follows

$$f(x) = \begin{cases} y_0, & \text{for } x \in A \cup D, \\ y_n, & \text{for } x \in A_n \cup ((G_n \setminus G_{n-1}) \cap B). \end{cases}$$

We shall show that $N_f = A$. We distinguish four cases.

I) Suppose that $x_0 \in A$. Then $f(x_0) = y_0$. Let U be an open neighborhood of x_0 and V an open neighborhood of $f(x_0)$. Then there is $k \in \mathbb{N}$ such that $y_n \in V$ for each $n > k$. Put $G = H_k \cap U$. Then G is an open neighborhood of x_0 and $G \subset U$. Since $G \cap G_n = \phi$ for each $n \leq k$, we have $G \setminus \bigcup_{n=1}^{k} A_n \subset G \cap f^{-1}(V)$. Since A_n are nowhere dense sets, we have $\mathrm{int}(G - \bigcup_{n=1}^{k} A_n) \neq \phi$. Hence $\phi \neq \mathrm{int}(G \cap f^{-1}(V)) = G \cap \mathrm{int}\, f^{-1}(V)$. Therefore $f^{-1}(V) \setminus \mathrm{int}\, f^{-1}(V)$ is not dense in U. Thus $x_0 \in N_f$.

II) Suppose that $x_o \in (G_k \setminus G_{k-1}) \cap B$ for some $k \in \mathbb{N}$. Put $U = X -\ \mathrm{cl}\, A$ and $V = Y \setminus \{y_0\}$. Then U is an open neighborhood of x_0 and V is an open neighborhood of $f(x_0) = y_k$. We have $f^{-1}(V) \cap U = B$. Since B is dense in U and $\mathrm{int}\, B = \phi$, the set $f^{-1}(V) -\ \mathrm{int}\, f^{-1}(V)$ is dense in U. Thus $x_0 \notin N_f$.

III) Suppose that $x_0 \in D$. Since $x_0 \in X \setminus \text{cl } A$, there is $k \in \mathbb{N}$ such that $x_0 \in G_k \setminus G_{k-1}$. Put $U = G_k$ and $V = Y \setminus \{y_1, y_2, \ldots, y_k\}$. Then U is an open neighborhood of x_0 and V is an open neighborhood of $f(x_0) = y_0$. Since $D \subset f^{-1}(V)$, the set $f^{-1}(V)$ is dense in U. Since $U \cap B$ is dense in U and $U \cap B \cap f^{-1}(V) = \phi$, we have $U \cap \text{ int } f^{-1}(V) = \phi$. Hence $f^{-1}(V) \setminus \text{int } f^{-1}(V)$ is dense in U. Thus $x_0 \notin N_f$.

IV) Suppose that $x_0 \in A_k$ for some $k \in \mathbb{N}$. Put $U = X \setminus \text{cl int } A$ and $V = Y \setminus \{y_0\}$. Since the set A is almost closed, we have $x_0 \in A_k \subset X \setminus A \subset U$. Therefore U is an open neighborhood of x_0 and V is an open neighborhood of $f(x_0) = y_k$. Since $f^{-1}(V) = B \cup (\text{cl } A \setminus A)$, $\text{int } f^{-1}(V) = \phi$ and $\text{cl } f^{-1}(V) = (X \setminus \text{cl } A) \cup (\text{cl } A - \text{ int } A) = \text{cl}(X \setminus A)$. So $U = X \setminus \text{cl int } A \subset \text{cl}(X \setminus A) = \text{ cl}(f^{-1}(V) \setminus \text{int } f^{-1}(V))$. Thus $x_0 \notin N_f$.

Theorem 3. *Let X be a perfectly normal space such that X^d is a resolvable space. Let Y be a metric space such that $Y^d \neq \phi$. Let us assume that (i) or (ii) is satisfied. Let $A \subset X$. Then there is a function $f : X \to Y$ such that $N_f = A$ if and only if the set A has the properties (1), (2) and (3).*

REMARK 4. Theorems 1 and 3 are true if instead of (i) or (ii) we require

(iii) X is a k-Baire space (see [5]) and Y is a metric space with weight (see [4, p. 27]) less than k.

REMARK 5. It was shown in [7] that a set A is Q_f for some f if and only if int cl $A \setminus A$ is first category, which is stronger than condition (3). Whereas the sets A_f are generally closed, and the sets C_f are generally G_δ sets, the sets Q_f and N_f don't even have to be Lebesque measurable. However, they must have the Baire property.

Theorem 4. *Let $f : X \to Y$, where X is a Baire space and Y is a separable metric space. Then the following three statements are equivalent:*

(u) $X \setminus N_f$ is a set of the first category in X,

(v) N_f is a dense set in X,

(w) f is cliquish.

PROOF. *(u) $\Rightarrow$ (v): Obvious.*

(v) ⇒ (w): We have $X \setminus A_f \subset (X \setminus N_f) \cup (N_f \setminus A_f) = (\mathrm{cl}\, N_f \setminus N_f) \cup (N_f \setminus A_f)$. *Therefore according to Theorem 1 and Proposition 2* $X \setminus A_f$ *is an open set of the first category and hence* $X \setminus A_f = \phi$.
(w) ⇒ (u): Follow's from Remark 2.

The Riemann function shows that the assumption (v) in Theorem 4 cannot be replaced by the assumption "$N_f = X$*".*

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