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Jozef Doboš, Katedra matematiky VŠT, Švermova 9, 040 01 Košice, Czechoslovakia

ON EXTREMAL VALUES OF CONTINUOUS MONOTONE FUNCTIONS

The notion of monotone for continuous functions was introduced by G.T. Whyburn [3]. The purpose of this note is to investigate extremal values of continuous monotone functions of several variables.

Definition. (Also see [2].) Let X be a topological space. A function $f : X \to R$ is said to be monotone if for each point $y \in R$, $f^{-1}(y)$ is connected in X.

Lemma. (See [1].) Let X be a locally connected space. Then $f : X \to R$ is continuous if and only if f has the Darboux property and there is a dense set $P \subset R$ such that $f^{-1}(p)$ is closed for each $p \in P$. (Recall that f is said to have the Dàrboux property if it maps connected sets to connected sets.)

Proposition. Let X be a locally connected space. Let $f : X \to R$ be a monotone function with the Darboux property. Then f is continuous.

Proof. Let $y \in R$. Suppose that $x \in Cl\, f^{-1}(y)$. Then $A = f^{-1}(y) \cup \{x\}$ is a connected set in X. Thus $f(A)$ is connected in R. Hence $f(x) = y$. Therefore $f^{-1}(y)$ is closed

Theorem. Let X be a T_3-space without isolated points. Suppose that for each $x \in X$ there is a base $\mathcal{B}(x)$ of open neighborhoods of x such that for each $B \in \mathcal{B}(x)$ the sets B, $X - B$ are connected and $Fr\, B$ is compact (where $Fr\, T = Cl\, T - Int\, T$). Let $f : X \to R$ be a monotone function with the Darboux property. Then f has a strict absolute extremum at any point a where f has a strict relative extremum.

Proof. Suppose f has a strict relative maximum at $a \in X$. (The second case is similar.) Let $b \in X$ such that $f(a) \leqq f(b)$. Then there is an open neighborhood U of a such that

(1) $$\forall \quad x \in U, \quad x \neq a : f(x) < f(a).$$

Since a is an accumulation point of X, there is $c \in U - \{a\}$. Since X is T_3, there is a closed neighborhood V of a such that $V \subset U - \{c\}$. By the assumption there is an open neighborhood A of a such that $A \subset V$, the sets A, $X - A$ are

connected and $Fr\,A$ is compact. Since a is an accumulation point of X, there is a $d \in A - \{a\}$.

Now we show that there is a real number r such that

$$\max(f(c), f(d)) < r < f(a), \tag{2}$$

$$f^{-1}(r) \cap Fr\,A = \emptyset. \tag{3}$$

Suppose that there is no such r. Let $\{r_n\}_{n=1}^{\infty}$ be a sequence of reals such that

$$\max(f(c), f(d)) < r_n < f(a) \quad (n = 1, 2, \ldots),$$

$$r_n \to f(a).$$

Then there are $x_n \in X$ such that $x_n \in f^{-1}(r_n) \cap Fr\,A \quad (n = 1, 2, \ldots)$. Since $Fr\,A$ is compact, there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $x_{n_k} \to x_0 \in Fr\,A$. From the continuity of f it follows that

$$r_{n_k} = f(x_{n_k}) \to f(x_0).$$

Since $r_{n_k} \to f(a)$, we have $f(a) = f(x_0)$, which contradicts (1).

Put $B = X - Cl\,A$. By (3) we obtain $f^{-1}(r) \subset A \cup B$. By (2) we have $f(d) < r < f(a)$. From the Darboux property of f it follows that there is $u \in A$ such that $f(u) = r$. Thus $A \cap f^{-1}(r) \neq \emptyset$. By (1) we obtain $f(c) < r < f(b)$. From the Darboux property of f it follows that there is $v \in X - A$ such that $f(v) = r$. By (3) we have $v \in X - Fr\,A$. Thus $B \cap f^{-1}(r) \neq \emptyset$. Since A, B are open, the set $f^{-1}(r)$ is not connected, which contradicts the assumption.

Corollary. No function $f: R^n \to R \quad (n \geqq 2)$ which is monotone and has the Darboux property can have relative extremal values.

The following example shows that some of the assumptions of the Theorem cannot be omitted.

Example. Put $X = [0, 1] \cap Q$. Define $f : X \to R$ as follows

$$f(x) = 2x + 3 - |2x - 1| + \sqrt{2}(1 - 2x - |2x - 1|) \quad (x \in X).$$

Then f is continuous and monotone. Evidently the point 0 is a strict relative minimum for f, but it is not a strict absolute minimum.

References

[1] Lipiński, J.S., *Une remarque sur la continuité et la connexité*, Coll. Math. 19, (1968), 251-253.

[2] Noiri, T., *Regular-closed functions and Hausdorff spaces*, Math. Nachr. 99, (1980), 217-219.

[3] Whyburn, G.T., *Non-alternating transformations*, Amer. J. of Math. 56, (1934), 294-302.

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