ON CERTAIN DECOMPOSITIONS OF CONTINUITY(*)

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SOMMARIO.- *Nell'articolo si introduce una nozione indebolita del concetto di continuità. Tale definizione (detta "mild continuity") generalizza vari tipi di continuità considerati precedentemente. Come risultato principale, si riesce a caratterizzare la continuità di una funzione usando forme più deboli di continuità.*

SUMMARY.- *In the paper the notion of mild continuity is introduced. It generalizes some well-known types of continuity (e.g. cliquishness, quasi-continuity, closedness of the graph). As a main result it is proved that a function is continuous if and only if it is mildly continuous and almost continuous.*

In the literature there are many papers dealing with almost continuity in connection with the decomposition of continuity. See for example [4], [13], where it is proved that a function is continuous if and only if it is almost continuous and cliquish, similarly, in [11], [12] continuity is decomposed into almost continuity and quasi-continuity and, in [8], into almost continuity and graph closedness.

In the present paper we give a simultaneous generalization of the decompositions mentioned above. We introduce a class of functions containing cliquish functions, quasi-continuous functions and functions with the closed graph. The corresponding generalized continuity (called mild continuity) combined with almost continuity gives a decomposition of continuity.

In what follows X, Y denote topological spaces. For a subset A of a topological space denote $Cl\,A$ and $Int\,A$ the closure and the interior of A, respectively. The letters Q and R stand for the set of rational and real numbers, respectively.

DEFINITION 1. (See [7].) A function $f : X \to Y$ is said to be almost continuous (known also as nearly continuous, see [11]) at a point $x \in X$ if

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for each neighbourhood V of $f(x)$, $Cl\, f^{-1}(V)$ is a neighbourhood of x. If f is almost continuous at every $x \in X$, then it is called almost continuous.

DEFINITION 2. (See [9].) A function $f : X \to Y$ is said to be quasi-continuous at a point $x \in X$ if for each neighbourhood U of x and each neighbourhood V of $f(x)$ there exists a nonempty open set $G \subset U$ such that $f(G) \subset V$. Denote by Q_f the set of all points at which f is quasi-continuous. If $Q_f = X$, then f is called quasi-continuous.

THEOREM A. (See [12], also [11].) *Let Y be a regular space. Then $f : X \to Y$ is continuous if and only if it is almost continuous and quasi-continuous.*

The following example shows that the assumption of the regularity of Y in Theorem A cannot be omitted.

EXAMPLE 1. Let $X = R$ with the usual topology. Let $Y = R$ with the topology $\mathcal{T} = \mathcal{F} \cup \{\emptyset\}$, where $\mathcal{F}$ is the filter consisting of all sets the complements of which are nowhere dense sets in usual topology of R. Let $f : X \to Y$ be the identity function. Then f is almost continuous and quasi-continuous, but it is not continuous.

The following example shows that Theorem A does not hold pointwise.

EXAMPLE 2. Let $f : R \to R$, $f(x) = 1$ for $x = 1/n$ ($n = 1, 2, \ldots$) and $f(x) = 0$ otherwise. Then f is quasi-continuous and almost continuous at the point 0, however it is not continuous at the point 0.

THEOREM 1. *Let Y be a regular space. Let $f : X \to Y$ be almost continuous. Then the set Q_f is closed in X.*

Proof. Let $x \in Cl\, Q_f$. Let U and V be open neighbourhoods of x and $f(x)$ respectively. Choose a neighbourhood W of $f(x)$ such that $Cl\, W \subset V$. From the almost continuity at x there is an open neighbourhood H of x such that the set $f^{-1}(W)$ is dense in H. Since $x \in Cl\, Q_f$, there is a point $y \in Q_f \cap H \cap U$. Let S be a neighbourhood of $f(y)$. From the quasi-continuity at y there is a nonempty open set $T \subset H \cap U$ such that $f(T) \subset S$. Since $f^{-1}(W)$ is dense in H, we have $f^{-1}(W) \cap T \neq \emptyset$. Then $\emptyset \neq W \cap f(T) \subset W \cap S$. Thus each neighbourhood S of $f(y)$ intersects the set W, which yields $f(y) \in Cl\, W \subset V$. Therefore V is a neighbourhood of $f(y)$. From the quasi-continuity at y there is a nonempty open set $G \subset U$ such that $f(G) \subset V$. Thus $x \in Q_f$.

The following example shows that the assumption of the regularity of Y in Theorem 1 cannot be omitted.

EXAMPLE 3. Let $X = R$ with the usual topology. Let $Y = \{a, b\}$, $\mathcal{T} = \{\emptyset, (b), Y\}$. Let $f : X \to Y, f(x) = a$ for $x \in Q, f(x) = b$ otherwise. Then f is almost continuous, Q_f is dense in X, however f is not quasi-continuous.

From Theorem A and Theorem 1 we obtain the following

THEOREM B. (See [6].) *Let Y be a regular space. Then $f : X \to Y$ is continuous if and only if it is almost continuous and the set Q_f is dense in X.*

DEFINITION 3. We say that a function $f : X \to Y$ is pointwise discontinuous if the set of all continuity points of f is dense in X.

COROLLARY 1. (See [1].) *Let Y be a regular space. Then $f : X \to$ Y* is continuous if and only if it is almost continuous and pointwise discontinuous.

The following example shows that a quasi-continuous real function need not be pointwise discontinuous.

EXAMPLE 4. Let $Q = \{q_1, q_2, q_3, ...\}$. Let $f : Q \to R, f(x) = \sum_{n:q_n<x} 2^{-n}$ for each $x \in Q$. Then f is a quasi-continuous function without points of continuity.

DEFINITION 4. (See [8].) We say that a function $f : X \to Y$ has the closed graph if the set $\{(x,y) \in X \times Y; y = f(x)\}$ is a closed subset of the product $X \times Y$.

COROLLARY 2. *Let X be a Baire space and Y be a σ-compact T_3 space. Then $f : X \to Y$ is continuous if and only if it is almost continuous and has the closed graph.*

Proof. Suppose that $f : X \to Y$ has the closed graph. Then by [3] the function f is pointwise continuous.

DEFINITION 5. Let $f: X \to Y$. Put $S_f = \{x \in X$; there is a base $\mathcal{A}$ of neighbourhoods of $f(x)$ such that for every $A \in \mathcal{A}$ and for every neighbourhood U of x the set $f^{-1}(A) - Int f^{-1}(A)$ is not dense in $U\}$. We say that f is mildly continuous, if the set S_f is dense in X.

LEMMA 1. *Let $f: X \to Y$. Then $Q_f \subset S_f$.*

Proof. Let $x \in Q_f$. Let U and V be neighbourhoods of x and $f(x)$ respectively. We shall show that $f^{-1}(V) - Int f^{-1}(V)$ is not dense in U. Since $x \in Q_f$, there is a nonempty open set $G \subset U$ such that $f(G) \subset V$. Thus $G \subset Int f^{-1}(V)$, which yields $(f^{-1}(V) - Int f^{-1}(V)) \cap G = \emptyset$.

LEMMA 2. *Let $f: X \to Y$ be almost continuous. Then $S_f \subset Q_f$.*

Proof. Let $x \in S_f$. Let U and V be neighbourhoods of x and $f(x)$ respectively. Let A be a neighbourhood of $f(x)$ such that $A \subset V$ and for each neighbourhood T of x the set $f^{-1}(A) - Int f^{-1}(A)$ is not dense in T. From the almost continuity at x there is a neighbourhood W of x such that $f^{-1}(A)$ is dense in W. Now there exists a nonempty open set $H \subset U \cap W$ such that $(f^{-1}(A) - Int f^{-1}(A)) \cap H = \emptyset$. Since $f^{-1}(A)$ is dense in W, we obtain $f^{-1}(A) \cap H \neq \emptyset$. Hence $Int f^{-1}(A) \cap H \neq \emptyset$. Put $G = Int f^{-1}(A) \cap H$. Then G is a nonempty open set such that $G \subset U$ and $f(G) \subset V$. Thus $x \in Q_f$.

We are now able to establish the main theorem, which follows from Lemmas 1 and 2 and Theorem B.

THEOREM 2. *Let Y be a regular space. Then $f: X \to Y$ is continuous if and only if it is almost continuous and mildly continuous.*

COROLLARY 3. (See [8].) *Let Y be a locally compact Hausdorff space. Then $f: X \to Y$ is continuous if and only if it is almost continuous and has the closed graph.*

Proof. Suppose that $f: X \to Y$ has the closed graph. Let $x \in X$. Let $\mathcal{A}$ be a base of compact neighbourhoods of the point $f(x)$. Since f has the closed graph, by [5] the set $f^{-1}(A)$ is closed in X for each $A \in \mathcal{A}$. Thus $f^{-1}(A) - Int f^{-1}(A)$ is nowhere dense in X. This shows that $x \in S_f$.

DEFINITION 6. (See [9].) Let Y be a metric space with a metric d. A function $f: X \to Y$ is said to be cliquish at a point $x \in X$ if for each $\varepsilon > 0$ and each neighbourhood U of the point x there exists a nonempty open set

$G \subset U$ such that $d(f(y), f(z)) < \varepsilon$ for each $y, z \in G$. Denote by A_f the set of all points at which $f: X \to Y$ is cliquish. If $A_f = X$, then f is called cliquish.

The following example shows that the cliquishness is not a topological notion.

EXAMPLE 5. Let $X = Q \cap (0, 1)$ with the usual topology. Put $Y = \{1/n; n = 1, 2, 3, \ldots\}$. Let d_1 be the usual metric on Y and d_2 the discrete metric on Y (i.e. $d_2(x, y) = 1$ for $x \neq y$). Then d_1 and d_2 are topologically equivalent. Let $f: X \to Y$ be the Riemann function (i.e. $f(x) = 1/q$ for $x = p/q$, where p, q are relatively prime integers, $q > 0$). Then $f: X \to (Y, d_1)$ is cliquish, while $f: X \to (Y, d_2)$ is not cliquish.

LEMMA 3. *Let Y be a metric space with a metric d. Let $f: X \to Y$. Then $A_f \subset S_f$.*

Proof. Let $x \in A_f - S_f$. Let $\varepsilon_0 > 0$ be a such that for each neighbourhood V of $f(x)$, $V \subset S(f(x), \varepsilon_0)$, there is a neighbourhood T of x such that the set $f^{-1}(V) - Int\, f^{-1}(V)$ is dense in T. Let $\varepsilon > 0$. We may assume $\varepsilon < \varepsilon_0$. Then there is a neighbourhood W of x such that the set $H = f^{-1}(S(f(x), \varepsilon/2)) - Int\, f^{-1}(S(f(x), \varepsilon/2))$ is dense in W. Let U be a neighbourhood of x. Since $x \in A_f$, there is a nonempty open set $G \subset U \cap W$ such that $d(f(y), f(z)) < \varepsilon/2$ for each $y, z \in G$. Since H is dense in W, we have $G \cap H \neq \emptyset$. Choose $y \in G \cap H$. Let $z \in G$. Then $d(f(x), f(z)) \leq d(f(x), f(y)) + d(f(y), f(z)) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Therefore $f(z) \in S(f(x), \varepsilon)$. Thus $f(G) \subset S(f(x), \varepsilon)$. Hence $x \in Q_f$, which contradicts to Lemma 1. Thus $A_f - S_f = \emptyset$.

COROLLARY 4. (See [13], also [4].) *Let Y be a metric space. Then $f: X \to Y$ is continuous if and only if it is almost continuous and cliquish.*

DEFINITION 7. (See [2].) A function $f: X \to Y$ is said to be simply continuous if for every open set V of Y, $f^{-1}(V)$ is the union of an open set of X and a nowhere dense set of X.

COROLLARY 5. *Let Y be a regular space. Then $f: X \to Y$ is continuous if and only if it is almost continuous and simply continuous.*

DEFINITION 8. (See [10].) A function $f: X \to Y$ is said to be barely continuous if $f_{|M}$ has a point of the continuity for each nonempty closed set M in X.

LEMMA 4. *Let Y be a regular space. If $f: X \to Y$ is barely continuous, then it is mildly continuous.*

Proof. Suppose that $f : X \to Y$ is barely continuous. Let $x \in X$. If $x \in Q_f$, by Lemma 1 we have $x \in S_f$. Suppose that $x \notin Q_f$. Then there is an open neighbourhood W of $f(x)$ and an open neighbourhood U of x such that for each nonempty open set $G \subset U$ there is $y \in G - f^{-1}(W)$. Let V be a neighbourhood of $f(x)$ such that $Cl\ V \subset W$. The set $\mathcal{A} = \{T \subset Y;\ T$ is a neighbourhood of $f(x)$ and $T \subset V\}$ is a base of neighbourhoods of $f(x)$. Let S be a neighbourhood of x and $T \in \mathcal{A}$. We shall show that $f^{-1}(T)$ - $Int\ f^{-1}(T)$ is not dense in S. Clearly $P = U \cap Int\ S$ is an open neighbourhood of x. Let z be a continuity point of $f_{|Cl\ P}$. We shall show that $f(z) \notin W$. If namely $f(z) \in W$, then W is a neighbourhood of $f(z)$ and hence there is an open neighbourhood A of z in X such that $f(A \cap Cl\ P) \subset W$. However, $A \cap P$ is a nonempty open subset of U and hence there is $y \in A \cap P$ with $f(y) \notin W$, a contradiction. Thus $f(z) \notin W$. This implies that $Y - Cl\ V$ is a neighbourhood of $f(z)$. Hence there is an open neighbourhood B of z in X such that $f(B \cap Cl\ P) \subset Y - Cl\ V$. Then $B \cap P$ is a nonempty open subset of S and $f(B \cap P) \subset Y - Cl\ V$. Therefore $(B \cap P) \cap f^{-1}(V) = \emptyset$, i. e. $f^{-1}(V)$ is not dense in S. Hence $f^{-1}(T)$ is not dense in S. Therefore $f^{-1}(T)$ – Int $f^{-1}(T)$ is not dense in S.

The assumption of the regularity of Y is essential, as the following example shows.

EXAMPLE 6. Let $X = Y = \{a, b, c\}$, $\mathcal{S} = \{\emptyset, \{a, c\}, X\}$, $\mathcal{T} = \{\emptyset, \{a\}, \{c\}, \{a, c\}, Y\}$. Let $f : (X, \mathcal{S}) \to (Y, \mathcal{T})$ be the identity function. Then f is barely continuous, but it is not mildly continuous.

COROLLARY 6. *Let Y be a regular space. Then $f : X \to Y$ is continuous if and only if it is almost continuous and barely continuous.*

The following example shows that the assumption of the mild continuity in Theorem 2 cannot be replaced by the assumption "first Baire class" or by the assumption that the set of all discontinuity points is of the first category.

EXAMPLE 7. Let A be a dense subset of Q such that $Q - A$ is dense in Q. Let $f : Q \to R$, $f(x) = 0$ for each $x \in A$ and $f(x) = 1$ otherwise. Then f is almost continuous of the first Baire class, but $S_f = \emptyset$.

It is well-known that the set of all points of the discontinuity of a quasi-continuous (cliquish, pointwise discontinuous, simply continuous, barely continuous, with the closed graph) real function of a real variable is of the first category. We shall show that there is a mildly continuous real function of a real variable without points of the continuity.

EXAMPLE 8. Let $f : R \to R$, $f(x) = q$ for $x = p/q$, where p, q are relatively prime integers, $q > 0$, $f(x) = 0$ otherwise. Then f is mildly continuous, nevertheless f is discontinuous at each point.

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