Math. Slovaca 37, 1987, No. 1, 3—7

ON STATIONARY SETS FOR CERTAIN GENERALIZATIONS OF CONTINUITY

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Let X, Y be two topological spaces. Let $\mathscr{F}$ be a class of functions $f\colon X \to Y$. A subset A of X with the property that whenever $f \in \mathscr{F}$ is constant on A, then f must be constant on X, is said to be a stationary set for $\mathscr{F}$. Observe that if $\mathscr{F}_1 \subset \mathscr{F}_2$, then each stationary set for $\mathscr{F}_2$ is also a stationary set for $\mathscr{F}_1$. (See [2], p. 199.)

For the basic properties of stationary sets see [1]. A survey of results of stationary sets for derivatives is in [3].

In the present paper we give a complete characterization of stationary sets for certain generalizations of continuity (for somewhat continuous functions and quasi-continuous functions).

Let X, Y be two topological spaces. A function $f\colon X \to Y$ is said to be somewhat continuous if for each set $V \subset Y$ open in Y such that $f^{-1}(V) \neq \emptyset$ there exists a nonempty open set $U \subset X$ so that $U \subset f^{-1}(V)$ (see [5]).

A function $f\colon X \to Y$ is said to be quasi-continuous at the point $x_0 \in X$ if for each neighbourhood $U(x_0)$ of the point x_0 (in X) and each neighbourhood $V(f(x_0))$ of the point $f(x_0)$ (in Y) there exists a nonempty open set $U \subset U(x_0)$ such that $f(U) \subset V(f(x_0))$. A function $f\colon X \to Y$ is said to be quasi-continuous on X if it is quasi-continuous at each point $x \in X$. (See [6] and [9].)

In the sequel $S(X, Y)$ and $Q(X, Y)$ denote the sets of all functions $f\colon X \to Y$ which are somewhat continuous and quasicontinuous on X, respectively.

In the paper it is supposed that Y is a Hausdorff space which has at least two elements.

1. Stationary sets for the class of somewhat continuous functions

In the following theorem we give a characterization of the family of all stationary sets for the class of somewhat continuous functions.

1.1. Theorem. *Let A be a nonempty subset of a topological space X. Then A is a stationary set for the class $S(X, Y)$ if and only if every nonempty open subset of the set $X - \mathrm{Cl}\, A$ is dense in X.*

Proof. Sufficiency. By contradiction. Let every nonempty open subset of the set $X - \mathrm{Cl}\, A$ be dense in X. Let $f \in S(X, Y)$, $f(x) = a$ for $x \in A$, $f(b) \neq a$ for some $b \in X - A$. Choose open disjoint neighbourhoods U and V of the points a and $f(b)$, respectively. Then $G = \mathrm{Int}\, f^{-1}(U)$ and $H = \mathrm{Int}\, f^{-1}(V)$ are nonempty and disjoint. Since $A \subset X - H$ and H is open, we have $H \subset X - \mathrm{Cl}\, A$. Then, by the assumption, H is dense in X. Therefore $G \cap H \neq \emptyset$, a contradiction.

Necessity. Deny. Suppose that there exists a nonempty open set $W \subset X - \mathrm{Cl}\, A$ which is not dense in X. Choose $u, v \in Y$ such that $u \neq v$. Define the function $f: X \to Y$ as follows

$$f(x) = \begin{cases} u & \text{for } x \in W, \\ v & \text{otherwise.} \end{cases}$$

Then f is somewhat continuous function which is constant on A, but f is not constant on X. Hence the set A is not stationary for the class $S(X, Y)$. The proof is complete.

1.2. Remark. Let X be a topological space. Then every dense subset of X is a stationary set for the class $S(X, Y)$.

1.3. Definition. An open almost-base for a space X is a family $\mathscr{A}$ of open subsets of X such that every nonempty open subset of X contains some nonempty $A \in \mathscr{A}$ (see [4]).

1.4. Theorem. *Let X be a topological space which is not antidiscrete. Then every stationary set for the class $S(X, Y)$ is dense in X if and only if the family of all open subsets of X which are not dense in X is an almost-base for X.*

Proof. Sufficiency. Deny. Let A be a stationary set for the class $S(X, Y)$ which is not dense in X. Evidently $X - \mathrm{Cl}\, A$ is a nonempty open set and by Theorem 1.1 we have that every nonempty open subset of the set $X - \mathrm{Cl}\, A$ is dense in X.

Necessity. Deny. Suppose that there exists a nonempty open set U in X such that every nonempty open subset of this set is dense in X. Let V be a nonempty open proper subset of X. Since U is dense in X, the set $W = U \cap V$ is nonempty. By Theorem 1.1 the set $X - W$ is stationary for the class $S(X, Y)$. But $X - W$ is not dense in X. The proof is complete.

By. 1.4 we have the following theorem.

1.5. Theorem. *Let A be a subset of a Hausdorff space X. Then A is a stationary set for the class $S(X, Y)$ if and only if A is dense in X.*

1.6. Definition. A space X is said to be hyperconnected if every nonempty open set is dense in X (see [7]).

1.7. Theorem. *Let X be a topological space. Then X is hyperconnected if and only if each somewhat continuous function $f: X \to Y$ is constant on X.*

Proof. It is sufficient to take into account the following reason. If X is not hyperconnected, then there is a set $A \subset X$ such that $\text{Int}\, A \neq \emptyset \neq \text{Int}\,(X - A)$.

1.8. Corollary. *Let X be a hyperconnected space. Then every nonempty subset of X is a stationary set for the class $S(X, Y)$.*

The following example shows that the assumption "Hausdorff space" in the Theorem 1.5 cannot be replaced by the assumption "T_1-space".

1.9. Example. Let X be an infinite countably set with the cofinite topology. Evidently X is a T_1-space. Since X is hyperconnected, by 1.8 each nonempty finite set is a stationary set for the class $S(X, Y)$, but it is not dense in X.

2. Stationary sets for the class of quasi-continuous functions

2.1. Definition. Let A be a subset of a topological space X. The set A is regular open if $A = \text{Int Cl}\, A$ (see [8]).

In the following theorem we give a characterization of the family of all stationary sets for the class of quasi-continuous functions.

2.2. Theorem. *Let A be a nonempty subset of a topological space X. Then A is a stationary set for the class $Q(X, Y)$* if *and only if the set $X - \text{Cl}\, A$ has not a nonempty regular open subset.*

Proof. Sufficiency. By contradiction. Suppose that the set $X - \text{Cl}\, A$ has not a nonempty regular open subset. Let $f \in Q(X, Y)$, $f(x) = a$ for $x \in A$, $f(b) \neq a$ for some $b \in X - A$. Choose open disjoint neighbourhoods U and V of the points a and $f(b)$, respectively. Since f is quasi-continuous at the point b, there exists a nonempty open set W in X such that $f(W) \subset V$. Since $\text{Int Cl}\, W$ is nonempty regular open, by the assumption we have $A \cap \text{Int Cl}\, W \neq \emptyset$. Choose a point c in this intersection. Since f is quasi-continuous at the point c, there exists a nonempty open set $G \subset \text{Int Cl}\, W$ such that $f(G) \subset U$. Since $G \subset Cl\, W$, we have $G \cap W \neq \emptyset$. Thus $\emptyset \neq f(G \cap W) \subset U \cap V = \emptyset$, a contradiction.

Necessity. Deny. Suppose that the set $X - \text{Cl}\, A$ has a nonempty regular open subset W. Choose $u, v \in Y$ such that $u \neq v$. Define the function $f: X \to Y$ as follows

$$f(x) = \begin{cases} u & \text{for } x \in W, \\ v & \text{otherwise.} \end{cases}$$

It is not difficult to verify that $f \in Q(X, Y)$. Evidently f is constant on A, but it is not constant on X. The proof is complete.

In the following we give some corollaries of this theorem.

2.3. Theorem. *Let X be a topological space. Then every stationary set for the class $Q(X, Y)$ is dense in X if and only if the family of all regular open sets is an almost-base for X.*

Proof. Sufficiency. Let A be a stationary set for the class $Q(X, Y)$. Then by Theorem 2.2 the set $X - \mathrm{Cl}\, A$ has not a nonempty regular open subset. Hence by hypothesis we have $X - \mathrm{Cl}\, A = \emptyset$. Thus A is dense in X.

Necessity. Let U be a nonempty open proper subset of X. Evidently the set $X - U$ is not dense in X, then by hypothesis $X - U$ is not stationary for the class $Q(X, Y)$. By Theorem 2.2 we have that the set $U = X - \mathrm{Cl}(X - U)$ has a nonempty regular open subset. Hence the desired property is ensured.

2.4. Theorem. *Let A be a subset of a regular space X. Then A is a stationary set for the class $Q(X, Y)$ if and only if A is dense in X.*

Proof. Let U be a nonempty open set. Take an arbitrary $u \in$ U. Since X is regular, there exists an open set V such that $u \in V \subset \mathrm{Cl}\, V \subset U$. Put $W = \mathrm{Int}\, \mathrm{Cl}\, V$. Then W is a nonempty regular open set such that $W \subset U$. By 2.3 every stationary set for $Q(X, Y)$ is dense. The converse is trivial.

The following example shows that the assumption "regular space" in Theorem 2.4 cannot be replaced by the assumption "Hausdorff space".

2.5. Example. Let X be the set of all real numbers. Denote by D the set of all rational numbers. Define a topology τ on X generated from the Euclidean topology on R by the addition of all sets of the form $D \cap U$ where U is an open set in the Euclidean topology on R. By Theorem 2.3 there exists a stationary set for the class $Q(X, Y)$ which is not dense in X, but X is a Hausdorff space.

The following example shows that there exists a Hausdorff space X such that the statement of Theorem 2.4 is true, but X is not regular.

2.6. Example. Let X be the set of all real numbers and $A = \{1/n; n = 1, 2, 3, \ldots\}$. Define a topology τ on X by letting $G \in \tau$ if $G = U - B$ where $B \subset A$ and U is an open set in the Euclidean topology on R. Then by Theorem 2.3 the desired property is ensured. But X is a Hausdorff space which is not regular.

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Received March 10, 1983

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ОБ СТАЦИОНАРНЫХ МНОЖЕСТВАХ ДЛЯ НЕКОТОРЫХ ОБОБЩЕНИЙ НЕПРЕРЫВНОСТИ

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Резюме

В настоящей работе мы предлагаем характеризацию стационарных множеств для некоторых обобщений непрерывности (для классов немножко-непрерывных функций и квазинепрерывных функций).