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ON QUASI-UNIFORM CONVERGENCE AND SYMMETRICALLY CONTINUOUS FUNCTIONS

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The function $f\colon R \to R$ is said to be locally symmetric at the point $x \in R$ if there exists a $\delta = \delta(x)$ such that for each h, $0 < h < \delta$, the equality $f(x-h) = = f(x+h)$ holds. The function $f\colon R \to R$ is said to be locally symmetric (on R) if it is locally symmetric at each point $x \in R$ (see [1]).

The function $f\colon R \to R$ is said to be symmetrically continuous at the point $x \in R$ if $\lim\limits_{h \to 0} (f(x+h) - f(x-h)) = 0$. The function $f\colon R \to R$ is said to be symmetrically continuous (on R) if it is symmetrically continuous at each point $x \in R$ (see [1]). Clearly each locally symmetric function is symmetrically continuous.

It is known (see [1]) that the class of all symmetrically continuous functions is closed with respect to uniform convergence. In connection with this fact the question arises whether the abovementioned class is closed with respect to quasi-uniform convergence. The purpose of the present note is to give a negative answer to this question.

For convenience we recall the definition of quasi-uniform convergence.

Definition. Let X be a set and Y a metric space (with the metric d). A sequence $\{f_n\}_{n=1}^{\infty}$ of functions $f_n\colon X \to Y$ $(n = 1, 2, \ldots)$ is said to be quasi-uniformly convergent to $f\colon X \to Y$ if

(i) $\{f_n\}_{n=1}^{\infty}$ converges pointwise to $f\colon X \to Y$, and

(ii) $\varepsilon > 0$ $m \in \{0, 1, 2, \ldots\}$ $p \in N$ $x \in X$:

$$\min \{d(f_{m+1}(x), f(x)), \ldots, d(f_{m+p}(x), f(x))\} < \varepsilon$$

(see [2], p. 143).

Lemma 1. Let X be a set and Y a metric space (with the metric d). Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions $f_n\colon X \to Y$ $(n = 1, 2, \ldots)$, converging pointwise to $f\colon X \to Y$. If there exist sets A, $B \subset X$ such that

$$A \cup B = X, \tag{1}$$

(2) for any $x \in A$ $k \in N$: $f_{2k}(x) = f(x)$,

(3) for any $x \in B$ $k \in N$: $f_{2k-1}(x) = f(x)$,

then $\{f_n\}_{n=1}^{\infty}$ converges quasi-uniformly to f: $X \to Y$.

Proof. Let $m \in \{0, 1, 2, \ldots\}$. Put $p = m + 2$. Then we have

$$\text{for each } x \in A: f_{m+p}(x) = f_{2m+2}(x) = f(x),$$

$$\text{for each } x \in B: f_{m+p-1}(x) = f_{2m+1}(x) = f(x).$$

Hence for each $x \in X$ we obtain

$$\min \{d(f_{m+1}(x), f(x)), \ldots, d(f_{m+p}(x), f(x))\} = 0 < \varepsilon.$$

Lemma 2. Let f, g: $R \to R$ be functions. Let us suppose that there exists a finite set $F \subset R$ such that $f(x) = g(x)$ for each $x \in R - F$. Then f is locally symmetric if and only if g is locally symmetric.

Proof. Let f be locally symmetric. Let $x \in R$. Then there exists $\delta_1 > 0$ such that

$$\text{for each } h \in R, |h| < \delta_1: f(x + h) = f(x - h).$$

Put $\delta = \min \{\delta_1, \text{dist}(x, F - \{x\})\} > 0$. Let $h \in R$, $0 < |h| < \delta$. Then $x + h$, $x - h \in R - F$, hence we have

$$g(x + h) = f(x + h) = f(x - h) = g(x - h).$$

Thus g is locally symmetric.

Lemma 3. Let f: $R \to R$ be locally symmetric at 0. Let for each $n \in N$ the set $\{x \in R - \langle -1/n, 1/n \rangle: f(x) \neq 0\}$ be finite. Then f is locally symmetric.

Proof. Let $x \in R$, $x \neq 0$. Let $n \in N$ such that $1/n < |x|/2$. Put $A = \{y \in R - \langle -1/n, 1/n \rangle: f(y) \neq 0\} - \{x\}$, $\delta = \min \{1/n, \text{dist}(x, A)\} > 0$. Let $h \in R$, $0 < |h| < \delta$. Since $|(x + h) - x| = |h| < \text{dist}(x, A)$, we obtain $x + h \notin A$. Since $|h| < 1/n$, we get

$$1/n < |x|/2 = |x| - |x|/2 < |x| - |h| \leqq |x + h|,$$

hence $x + h \notin \langle -1/n, 1/n \rangle$. Since $|h| > 0$, we have $x + h \notin \{x\}$. Then $x + h \in \{y \in R: f(y) = 0\}$, i.e. $f(x + h) = 0$. Thus f is locally symmetric at x.

Theorem. There exists a sequence $\{f_n\}_{n=1}^{\infty}$ of functions f_n: $R \to R$ $(n = 1, 2, \ldots)$ converging quasi-uniformly to f: $R \to R$ such that each f_n is locally symmetric but f is not symmetrically continuous.

Proof. Put $K = \{-1/(2k): k \in N\}$ and $L = \{1/(2k + 1): k \in N\}$. For each $n \in N$ put

$$K_n = K \cup \{1/(2k + 1): k < n\} \cup \{1/(2k): k \geqq n\},$$

$$L_n = L \cup \{-1/(2k): k < n\} \cup \{-1/(2k + 1): k \geqq n\}.$$

Define a sequence $\{f_n\}_{n=1}^{\infty}$ of functions f_n: $R \to R$ ($n = 1, 2, \ldots$) as follows

$$f_{2m}(x) = \begin{cases} 1 & \text{for } x \in K_m, \\ 0 & \text{for } x \in R - K_m, \end{cases} \qquad f_{2m-1}(x) = \begin{cases} 1 & \text{for } x \in L_m, \\ 0 & \text{for } x \in R - L_m, \end{cases}$$

($m = 1, 2, \ldots$). Define a function f: $R \to R$ as follows

$$f(x) = \begin{cases} 1 & \text{for } x \in K \cup L, \\ 0 & \text{otherwise.} \end{cases}$$

It is not difficult to verify that the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f. Put

$$A = (-\infty, 0\rangle \quad \text{and} \quad B = \langle 0, \infty).$$

Then by Lemma 1 the sequence $\{f_n\}_{n=1}^{\infty}$ converges quasi-uniformly to f. By Lemma 3 the functions f_1 and f_2 are locally symmetric. Hence by Lemma 2 each function f_n ($n = 1, 2, \ldots$) is locally symmetric. Since

$$\lim_{n \to \infty} (f(0 + 1/(2n + 1)) - f(0 - 1/(2n + 1))) = 1,$$

the function f is not symmetrically continuous.

REFERENCES

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[2] Sikorski, R.: Funkcje rzeczywiste I., PWN, Warszawa, 1958.

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SÚHRN

O KVÁZIROVNOMERNEJ KONVERGENCII A SYMETRICKY SPOJITÝCH FUNKCIÁCH

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V tejto práci dokazujeme, že množina všetkých symetricky spojitých funkcií nie je uzavretá vzhľadom na kvázirovnomernú konvergenciu.

РЕЗЮМЕ

ОБ КВАЗИРАВНОМЕРНОЙ СХОДИМОСТИ И СИММЕТРИЧЕСКИ НЕПРЕРЫВНЫХ ФУНКЦИЯХ

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В настоящей работе мы показываем, что множество всех симметрически непрерывных функций не является замкнутым относительно квазиравномерной сходимости.