

ON THE SET OF POINTS OF DISCONTINUITY FOR FUNCTIONS WITH CLOSED GRAPHS

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For two topological spaces X and Y and any function $f : X \rightarrow Y$ the subset $\{(x, f(x)); x \in X\}$ of the space $X \times Y$ (with Tychonoff's topology) is called the graph of f and is denoted by $G(f)$. We denote by $C_f(D_f)$ the set of all such points at which the function f defined on X is continuous (discontinuous).

I. Baggs [1] dealt with the set of points of discontinuity of functions with closed graphs. In this paper we shall generalize some results of the paper [1].

1. PRELIMINARIES

First we recall definitions and some basic properties.

Proposition A. *Let a function $f : X \rightarrow Y$ have a closed graph. If K is a compact subset of Y then $f^{-1}(K)$ is a closed subset of X . (See [2; Theorem 3.6].)*

This proposition has the following corollary.

Proposition B. *Let a function $f : X \rightarrow Y$ have a closed graph. Then $f^{-1}(y)$ is a closed subset of X for each $y \in Y$. (See [7; Theorem 1].)*

Proposition C. *Let $f : X \rightarrow Y$ be any function where Y is a locally compact Hausdorff space. If for each compact $K \subset Y$, $f^{-1}(K)$ is closed, then $G(f)$ is closed. (See [9; Theorem 6].)*

Definition 1. Let X and Y be topological spaces, let $f : X \rightarrow Y$ be a function and let $p \in X$. Then f is said to be *c-continuous at p* provided the following holds: if U is an open subset of Y containing $f(p)$ such that $Y - U$ is compact, then there is an open subset V of X containing p such that $f(V) \subset U$. The function f is said to be *c-continuous (on X)* provided f is c-continuous at each point of X . (See [3; Definition 1].)

Proposition D. Let $f : X \rightarrow Y$ be any function where Y is Hausdorff. Then the following statements are equivalent:

- (1) f is c -continuous, and
- (2) if K is a compact subset of Y , then $f^{-1}(K)$ is a closed subset of X . (See [3; Theorem 1].)

Corollary 1. Let $f : X \rightarrow Y$ be any function where Y is a locally compact Hausdorff space. Then f is c -continuous if and only if $G(f)$ is closed.

Definition 2. A function $f : X \rightarrow Y$ is *locally bounded* at $x_0 \in X$ if and only if there exists a compact subset K of Y such that $x_0 \in \text{Int}(f^{-1}(K))$. We denote by B_f the set of all such points at which the function f is locally bounded.

Lemma A. Let $f : X \rightarrow Y$ be given. Then $G(f)$ is closed if and only if for each $x \in X$ and $y \in Y$, where $y \neq f(x)$, there exist open sets U and V containing x and y , respectively, such that $f(U) \cap V = \emptyset$. (See [8; Lemma].)

Theorem 1. Let $f : X \rightarrow Y$ be given. If $G(f)$ is closed, then

$$B_f \subset C_f.$$

Proof. We may assume that Y has at least two elements (in the opposite case we evidently have $B_f = C_f$). Let the set $G(f)$ be closed. Let $x_0 \in B_f$. By Definition 2 there exists a compact set K (in Y) such that $x_0 \in \text{Int}(f^{-1}(K))$. Let V be an open neighbourhood of the point $f(x_0)$. Since $K - V$ is compact and $G(f)$ is closed, $f^{-1}(K - V)$ is closed by Proposition A. Put

$$U = \text{Int}(f^{-1}(K)) - f^{-1}(K - V).$$

Evidently U is an open neighbourhood of the point x_0 . We shall prove that $f(U) \subset V$. Let $x \in U$. Since $f(x) \in K$ and $f(x) \notin K - V$, evidently $f(x) \in V$. Hence $x_0 \in C_f$.

Corollary 2. Let $f : X \rightarrow Y$ be any function where Y is a locally compact space. If $G(f)$ is closed, then $B_f = C_f$.

The converse to Corollary 2 is not necessarily true as the following example shows.

Example 1. Let $X = Y = R$ (where R denotes the set of all real numbers) with the usual topology. Define a function $f : X \rightarrow Y$ as follows:

$$f(x) = \begin{cases} \frac{1}{x} \sin \frac{1}{x} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

Then $G(f)$ is not closed, but Y is locally compact and $B_f = C_f$.

Lemma 1. *Let the function $f : X \rightarrow Y$ have a closed graph. If K is a compact subset of Y then $f^{-1}(K) - B_f$ is nowhere dense in X .*

Proof. Let K be a compact subset of Y . Put

$$A = f^{-1}(K) - B_f.$$

By Definition 2 it is easy to see that B_f is open. By Proposition A the set $f^{-1}(K)$ is closed. Therefore A is closed. Now we shall prove that $\text{Int}(A) = \emptyset$. Let $x \in \text{Int}(A)$. Then A is a neighbourhood of the point x such that $f(A) \subset K$. Since K is compact by Definition 2 we have $x \in B_f$. This leads to a contradiction because $x \in A \subset X - B_f$. Therefore A is nowhere dense in X .

Theorem 2. *Let $f : X \rightarrow Y$ be any function where Y is a σ -compact space (i.e. Y is the countable union of compact sets). If $G(f)$ is closed, then $X - B_f$ is closed and of the first category (in X).*

Proof. By the assumption, $Y = \bigcup_{n=1}^{\infty} K_n$, where each K_n is compact. Let $n \in N$ (where N denotes the set of all positive integers). Put

$$A_n = f^{-1}(K_n) - B_f.$$

By Lemma 1 the set A_n is nowhere dense in X . Hence $X - B_f = \bigcup_{n=1}^{\infty} A_n$ is of the first category in X .

2. REAL FUNCTIONS WITH CLOSED GRAPHS

Let X be a topological space. Denote by $U(X)$ the class of all real functions defined on X with closed graphs.

From Corollary 2, Lemma 1, Proposition B and Theorem 2 we obtain the following three theorems.

Theorem 3. *Let $f \in U(X)$. Then the set D_f is closed and of the first category (in X).*

Theorem 4. *Let $f \in U(X)$. Then $D_f \cap f^{-1}(0)$ is closed and nowhere dense (in X).*

Theorem A. *Let X be a T_2 Baire space. If $f : X \rightarrow R^n$ (R^n - the Euclidean n -space) has a closed graph, then D_f is closed and nowhere dense in X . (See [1], and for metric spaces see [5; Theorems 4 and 5].)*

Theorem B. *A set $F \subset R$ is closed and nowhere dense if and only if there exists a function $f : R \rightarrow R$ such that f has a closed graph and $D_f = F$. (See [1].)*

Theorem 5. Let F be a closed, G_δ and nowhere dense subset of a normal topological space X . Let $u : X \rightarrow \langle 0, 1 \rangle$ be a continuous function such that $u^{-1}(0) = F$. Define a function $g : X \rightarrow R$ as follows:

$$g(x) = \begin{cases} 1/u(x) & \text{for } x \in X - F, \\ 0 & \text{for } x \in F. \end{cases}$$

Then g has a closed graph and $D_g = F$.

Proof. We show that the graph of g is closed. Let $\{(x_\alpha, g(x_\alpha))\}_{\alpha \in A}$ be a convergent net of points of the graph of g , i.e. there exist x_0 and y_0 such that $(x_\alpha, g(x_\alpha)) \rightarrow (x_0, y_0) \in X \times R$. We distinguish two cases.

a. Let there exist α_0 such that for every $\alpha > \alpha_0$ we have $x_\alpha \in F$. Since $x_\alpha \rightarrow x_0$ and F is closed, we obtain $x_0 \in F$. Hence $g(x_0) = 0 = y_0$.

b. For each α let there exist $\beta > \alpha$ such that $x_\beta \notin F$. It follows from the definition of g that $g(x) \geq 1$ whenever $x \in X - F$. The convergence of the net $\{g(x_\alpha)\}_{\alpha \in A}$ implies that there is α_0 such that for every $\alpha > \alpha_0$ we have $x_\alpha \in X - F$. Since u is continuous at the point x_0 and $g(x_\alpha) \rightarrow y_0$, we obtain $x_0 \in X - F$. Since g is continuous on the set $X - F$, it is not difficult to verify that $g(x_\alpha) \rightarrow g(x_0)$. Hence $g(x_0) = y_0$.

Finally, we show that $D_g = F$. Evidently g is continuous on the set $X - F$. Let $x \in F$. Because the set F is nowhere dense, we have $\omega_g(x) \geq 1$ for the oscillation of g in x . Hence $x \in D_g$. The following example shows that there exists a metric space X and a function $f \in U(X)$ such that D_f is not nowhere dense.

Example 2. Let $X = \{x_1, x_2, \dots\}$ be a countably dense subset of R . Define a function $f : X \rightarrow R$ as follows:

$$f(x_n) = n \quad (n = 1, 2, \dots).$$

Then f has a closed graph, but $D_f = X$ is not nowhere dense in X .

Proposition 1. Let X be a topological space. Let $f \in U(X)$. Then $|f| \in U(X)$.

Proof. Let $x_0 \in X$. Let $y \neq |f(x_0)|$. First suppose that $y \geq 0$. Since $y \neq f(x_0)$, by Lemma A there exist $\delta_1 > 0$ and a neighbourhood U_1 of the point x_0 such that

$$f(U_1) \cap (y - \delta_1, y + \delta_1) = \emptyset.$$

Since $-y \neq f(x_0)$, by Lemma A there exist $\delta_2 > 0$ and neighbourhood U_2 of the point x_0 such that

$$f(U_2) \cap (-y - \delta_2, -y + \delta_2) = \emptyset.$$

Put

$$U = U_1 \cap U_2,$$

$$\delta = \min(\delta_1, \delta_2),$$

$$V = (y - \delta, y + \delta).$$

Let $x \in U$. If $f(x) \geq 0$, since $f(x) \notin (y - \delta_1, y + \delta_1)$, we have $||f(x)| - y| = |f(x) - y| \geq \delta_1 \geq \delta$. Therefore $|f(x)| \notin V$. If $f(x) < 0$, then $f(x) \notin (-y - \delta_2, -y + \delta_2)$, hence $|y - |f(x)|| = |y + f(x)| \geq \delta_2 \geq \delta$. Therefore $|f(x)| \notin V$. In the case $y < 0$ put $U = X$, $V = (-\infty; 0)$. Then by Lemma A the function f has a closed graph.

Proposition 2. *Let X be a topological space. Let α be a real number. Let $f \in U(X)$. Then $\alpha \cdot f \in U(X)$.*

Proof. It is obvious that for $\alpha = 0$ we have $\alpha \cdot f \in U(X)$. Suppose that $\alpha \neq 0$. Let $x_0 \in X$. Let K be a compact subset of R such that $\alpha \cdot f(x_0) \notin K$. Since K is closed, there exists $\varepsilon > 0$ such that

$$(3) \quad (\alpha \cdot f(x_0) - \varepsilon, \alpha \cdot f(x_0) + \varepsilon) \cap K = \emptyset.$$

Let $k > 0$ be a bound of the set K (i.e., $K \subset \langle -k, k \rangle$). Put

$$h = \max(k, \varepsilon/|\alpha|),$$

$$K_1 = \langle -h, h \rangle - (f(x_0) - \varepsilon/|\alpha|, f(x_0) + \varepsilon/|\alpha|).$$

Since $f(x_0) \notin K_1$ and K_1 is compact, there exists a neighbourhood U of the point x_0 such that

$$(4) \quad f(U) \cap K_1 = \emptyset.$$

Let $x \in U$. If $\alpha \cdot f(x) \notin \langle -k, k \rangle$, evidently $\alpha \cdot f(x) \notin K$. Let $\alpha \cdot f(x) \in \langle -k, k \rangle$. Then $f(x) \in \langle -h, h \rangle$, therefore by (4) we have $f(x) \in (f(x_0) - \varepsilon/|\alpha|, f(x_0) + \varepsilon/|\alpha|)$. Thus $|\alpha \cdot f(x) - \alpha \cdot f(x_0)| = |\alpha| \cdot |f(x) - f(x_0)| < |\alpha| \cdot \varepsilon/|\alpha| = \varepsilon$, hence by (3) we have $\alpha \cdot f(x) \notin K$. Then Corollary 1 yields $\alpha \cdot f \in U(X)$.

Remark 1. Propositions 1 and 2 are proved in the paper [6] for X a metric space.

It is known that the class $U(X)$ is not closed with respect to addition (see [6; Example 3]). We prove that $U(X)$ is closed with respect to addition of nonnegative functions.

Theorem 6. *Let X be a topological space. Let $f, g \in U(X)$ be nonnegative functions. Then $f + g \in U(X)$.*

Proof. Let $x_0 \in X$. Let K be a compact subset of R such that $f(x_0) + g(x_0) \notin K$. The closedness of the set K implies that there exists $\varepsilon > 0$ such that

$$(5) \quad (f(x_0) + g(x_0) - \varepsilon, f(x_0) + g(x_0) + \varepsilon) \cap K = \emptyset.$$

Let $k > 0$ be a bound of the set K (i.e. $K \subset \langle -k, k \rangle$). Put

$$K_1 = \langle 0, k \rangle - (f(x_0) - \varepsilon/2, f(x_0) + \varepsilon/2),$$

$$K_2 = \langle 0, k \rangle - (g(x_0) - \varepsilon/2, g(x_0) + \varepsilon/2).$$

Since $f \in U(X)$, by Corollary 1 there exists a neighbourhood U_1 of the point x_0 such that

$$(6) \quad f(U_1) \cap K_1 = \emptyset.$$

Since $g \in U(X)$, by Corollary 1 there exists a neighbourhood U_2 of the point x_0 such that

$$(7) \quad g(U_2) \cap K_2 = \emptyset.$$

Put

$$U = U_1 \cap U_2.$$

Let $x \in U$. If $f(x) + g(x) > k$, evidently $f(x) + g(x) \notin K$. Let $f(x) + g(x) \in \langle 0, k \rangle$. Since by (6) we have $f(x) \in \langle 0, k \rangle - K_1$, by the definition of K_1 we obtain

$$(8) \quad f(x) \in (f(x_0) - \varepsilon/2, f(x_0) + \varepsilon/2).$$

Since by (7) we have $g(x) \in \langle 0, k \rangle - K_2$, by the definition of K_2 we obtain

$$(9) \quad g(x) \in (g(x_0) - \varepsilon/2, g(x_0) + \varepsilon/2).$$

From (8) and (9) it follows that $|(f(x) + g(x)) - (f(x_0) + g(x_0))| \leq |f(x) - f(x_0)| + |g(x) - g(x_0)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$, hence by (5) we have $f(x) + g(x) \notin K$. Therefore $(f + g)(U) \cap K = \emptyset$. By Corollary 1 we obtain $f + g \in U(X)$.

Corollary 3. Let X be a topological space. Let $f, g \in U(X)$. Then $|f| + |g| \in U(X)$.

Definition 3. A topological space X is called *perfectly normal* if and only if it is normal and each closed subset of X is G_δ . (See [4], p. 181.)

Theorem 7. Let X be a perfectly normal topological space. Then $A \subset X$ is closed and of the first category in X if and only if there exists a function $f \in U(X)$ such that $D_f = A$.

Proof. *Necessity* follows from Theorem 3. *Sufficiency.* Let $A \subset X$ be closed and of the first category in X . Then $A = \bigcup_{n=1}^{\infty} A_n$, where each A_n is a closed nowhere dense subset of X , $A_n \subset A_{n+1}$ ($n = 1, 2, \dots$). Let $g : X \rightarrow \langle 0, 1 \rangle$ be a continuous function such that $g^{-1}(0) = A$. Let $g_n : X \rightarrow \langle 0, 1 \rangle$ ($n = 1, 2, \dots$) be continuous functions such that for each $n \in N$

$$(10) \quad g_n^{-1}(0) = A,$$

$$(11) \quad g_n(x) \geq g(x) \quad \text{for each } x \in X.$$

The existence of functions g, g_n ($n = 1, 2, \dots$) follows from Urysohn's lemma. For each $n \in N$ define a function $f_n : X \rightarrow R$ as follows:

$$f_n(x) = \begin{cases} 1/g_n(x) & \text{for } x \in X - A_n, \\ 0 & \text{for } x \in A_n. \end{cases}$$

By Theorem 5 we have $f_n \in U(X)$ ($n = 1, 2, \dots$). Consider the series

$$(12) \quad \sum_{n=1}^{\infty} ((1/2^n) \cdot f_n).$$

We now show that the series (12) is convergent to some function $f : X \rightarrow R$. If $x \in A_m$ for some $m \in N$, then

$$0 \leq \sum_{n=1}^{\infty} ((1/2^n) \cdot f_n(x)) = \sum_{n=1}^m ((1/2^n) \cdot f_n(x)) < +\infty.$$

If $x \in X - A$, then

$$\begin{aligned} 0 &\leq \sum_{n=1}^{\infty} ((1/2^n) \cdot f_n(x)) = \sum_{n=1}^{\infty} ((1/2^n) \cdot (1/g_n(x))) \leq \\ &\leq \sum_{n=1}^{\infty} ((1/2^n) \cdot (1/g(x))) = 1/g(x) < +\infty. \end{aligned}$$

We now show that $D_f = A$. First we shall prove that $X - A \subset C_f$. Let $b \in X - A$. Since $g(b) > 0$ and g is continuous at the point b , there exists a neighbourhood U of the point b such that

$$(13) \quad \forall x \in U : g(x) > g(b)/2.$$

Evidently $U \subset X - A$. Hence by (13) we have for each $x \in U$

$$f_n(x) = 1/g_n(x) \leq 1/g(x) < 2/g(b) \quad (n = 1, 2, \dots).$$

Therefore the series (12) is uniformly convergent on the set U . Since all functions f_n are continuous on U , the function f is continuous at the point b . Now we show that $A \subset D_f$. Let $a \in A$. Then $a \in A_m$ for some $m \in N$. We shall prove that for each neighbourhood V of the point a and for each $n \in N$ there exists a point $y \in V$ such that $f(y) > n$. Let V be a neighbourhood of the point a . Let $n \in N$. Since g_m is continuous at the point a , there exists a neighbourhood W of the point a such that

$$(14) \quad \forall x \in W : g_m(x) < 2^{-m}/n.$$

Since A_m is closed and nowhere dense, there exists a point $y \in V \cap W$ such that $y \in X - A_m$. Hence by (14) we have

$$f(y) \geq (1/2^m) \cdot (1/g_m(y)) > n.$$

Therefore $a \in X - B_f = D_f$.

Now we shall prove that $f \in U(X)$. Let K be a compact subset of R . We now show that $X - f^{-1}(K)$ is open. Let $x_1 \in A_m - f^{-1}(K)$ for some $m \in N$. Put

$$h = \sum_{n=1}^m ((1/2^n) \cdot f_n).$$

By Theorem 5, Proposition 2 and Theorem 6 we obtain $h \in U(X)$. Since $f(x_1) \notin K$ and K is closed, there exists $\varepsilon > 0$ such that

$$(f(x_1) - \varepsilon, f(x_1) + \varepsilon) \cap K = \emptyset.$$

Let $k > 0$ be a bound of the set K (i.e. $K \subset \langle -k, k \rangle$). Put

$$K_1 = \langle 0, k \rangle - (f(x_1) - \varepsilon, f(x_1) + \varepsilon).$$

Since $h(x_1) = f(x_1) \notin K_1$, K_1 is compact and $h \in U(X)$, by Proposition A the set $X - h^{-1}(K_1)$ is an open neighbourhood of the point x_1 . Since g_m is continuous at the point x_1 , there exists a neighbourhood U_1 of the point x_1 such that $U_1 \subset X - h^{-1}(K_1)$ and for each $x \in U_1$ we have

$$(15) \quad g_m(x) < 2^{-m}/(f(x_1) + \varepsilon).$$

If $x \in U_1 \cap A_m$, then $f(x) = h(x) \notin K_1$. Therefore $f(x) \notin K$. If $x \in U_1 - A_m$, then by (15) we have

$$h(x) \geq (1/2^m) \cdot f_m(x) = (1/2^m) \cdot (1/g_m(x)) > f(x_1) + \varepsilon.$$

Since $h(x) \notin K_1$, we obtain $h(x) \notin \langle 0, k \rangle$. Hence $f(x) \geq h(x) > k$, then $f(x) \notin K$. Therefore the point x_1 has a neighbourhood U_1 such that $U_1 \subset X - f^{-1}(K)$.

Let $x_2 \in (X - A) - f^{-1}(K)$. Since $x_2 \in C_f$, the set

$$U_2 = X - f^{-1}(K) = f^{-1}(R - K)$$

is a neighbourhood of the point x_2 .

Therefore the set $X - f^{-1}(K)$ is open. By Proposition C we have $f \in U(X)$.

This theorem has the following corollary.

Theorem C. *Let X be a Baire metric space. Then $F \subset X$ is closed and nowhere dense in X if and only if there exists a function $f \in U(X)$ such that $D_f = F$.*

The following example shows that the assumption “ X is perfectly normal” in Theorem 7 cannot be replaced by the assumption “ X is normal”.

Example 3. Let $X = \{\omega; \omega \leq \Omega\}$ (where Ω denotes the first uncountable ordinal number) with the order topology. It is well known that X is a normal space, and the set $\{\Omega\}$ is closed and nowhere dense in X but for each $f \in U(X)$ we have $D_f \neq \{\Omega\}$. (See [1].)

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