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ON METRIZATION OF THE UNIFORMITY OF A PRODUCT OF METRIC SPACES

JÁN BORSÍK and JOZEF DOBOŠ

Let T be a nonempty set. Denote by $\mathcal{M}(T)$ the set of all mappings $f: \{x \in R^T; \forall t \in T: x(t) \geqq 0\} \to R$ such that $d(x, y) = f(\{d_t(x(t), y(t))\}_{t \in T})$ is a metric on the set $\prod_{t \in T} M_t$ for every collection of metric spaces $\{(M_t, d_t)\}_{t \in T}$.

In [3] we have established a necessary and sufficient condition for the product topology on $\prod_{t \in T} M_t$ to be metrized by d. A natural question arises whether we can investigate metrizability by the metric d of the product uniformity on $\prod_{t \in T} M_t$. The special case when the index set T has exactly one element was solved in [2]. The present paper gives a complete answer regarding any index set T. The necessary and sufficient condition is formulated in Theorem.

For elements of the uniform spaces theory we refer to [1].

Definition 1. *Let $D = \{(M_t, d_t)\}_{t \in T}$ be a collection of metric spaces. Define a mapping $\varrho_D: \left(\prod_{t \in T} M_t\right)^2 \to R^T$ by*

$$(\varrho_D(x, y))(t) = d_t(x(t), y(t)) \tag{1}$$

for each $x, y \in \prod_{t \in T} M_t$, $t \in T$.

Definition 2. *Let T be a nonempty set. Suppose R^T to be ordered coordinate-wise, i.e.*

$$x \leqq y \quad \text{if and only if} \quad x(t) \leqq y(t) \quad \text{for each } t \in T.$$

Define a function $\Theta: T \to R$ by $\Theta(t) = 0$ for each $t \in T$. Put $T^+ = \{x \in R^T: x \geqq \Theta\}$. Denote by $\mathcal{M}(T)$ the set of all functions $f: T^+ \to R$ such that $f \circ \varrho_D$ is a metric for every collection of metric spaces $D = \{(M_t, d_t)\}_{t \in T}$.

Lemma 1. *Let $f \in \mathcal{M}(T)$. Then*

$$\forall x, y \in T^+: x \leqq 2y \Rightarrow f(x) \leqq 2 \cdot f(y), \tag{2}$$

(3) $$\forall x \in T^+ : f(x) = 0 \Leftrightarrow x = \Theta .$$

Proof. See [3].

Definition 3. *Let (M, d) be a metric space. Define a uniformity $\mathcal{U}$ on the set M as follows:*

(4) $$\mathcal{U} = \{A \subset M^2 ; \exists \varepsilon > 0 : d^{-1}(\langle 0, \varepsilon)) \subset A\} .$$

Definition 4. *Let $\{(M_t, d_t)\}_{t \in T}$ be a collection of metric spaces. Let $\{(M_t, \mathcal{U}_t)\}_{t \in T}$ be a collection of uniform spaces defined according to (4). Denote by $\mathcal{U}_D$ the product uniformity of the collection $\{(M_t, \mathcal{U}_t)\}_{t \in T}$, i.e.*

(5) $$\mathcal{U}_D = \{A \subset \Big(\prod_{t \in T} M_t\Big)^2 : \exists F \subset T, F \neq \emptyset \text{ finite } \forall t \in F\ \exists U_t \in \mathcal{U}_t :$$
$$\bigcap_{t \in F} (\pi_t \times \pi_t)^{-1}(U_t) \subset A\} ,$$

where π_t is the projection.

Denote by $\mathcal{U}_f$ the uniformity on the set $\prod_{t \in T} M_t$ derived by (4) from the metric $f \circ \varrho_D$.

Lemma 2. *Let $D = \{(M_t, d_t)\}_{t \in T}$ be a collection of metric spaces. Let $f \in \mathcal{M}(T)$. Then $\mathcal{U}_D \subset \mathcal{U}_f$.*

Proof. Let $U \in \mathcal{U}_D$. Then by (5) we have $\exists F \subset T$, $F \neq \emptyset$ *finite* $\forall t \in F\ \exists U_t \in \mathcal{U}_t$: $\bigcap_{t \in F} (\pi_t \times \pi_t)^{-1}(U_t) \subset U$. Let $t \in F$. Since $U_t \in \mathcal{U}_t$, there is, according to (4), a positive ε_t such that

$$d_t^{-1}(\langle 0, \varepsilon_t)) \subset U_t .$$

Denote $V_t = (\pi_t \times \pi_t)^{-1}(d_t^{-1}(\langle 0, \varepsilon_t)))$. Then we have

(6) $$\bigcap_{t \in F} V_t \subset \bigcap_{t \in F} (\pi_t \times \pi_t)^{-1}(U_t) \subset U .$$

Define a mapping $A : F \to R^T$ as follows:

$$(A(t))\,(i) = \begin{cases} 2\varepsilon, & \text{if } i = t \\ 0, & \text{if } i \in T - \{t\} . \end{cases}$$

for each $t \in F$, $i \in T$.

Let $t \in F$. Denote $\delta_t = f(A(t))/2$ (according to (3) we have $\delta_t > 0$) and $W_t = (f \circ \varrho_D)^{-1}(\langle 0, \delta_t))$.

Then $W_t \in \mathcal{U}_f$ for all $t \in F$, therefore

(7) $$\bigcap_{t \in F} W_t \in \mathcal{U}_f .$$

We show that $W_t \subset V_t$ for all $t \in F$. Let $t \in F$, $(x, y) \in W_t$. Then $f(\varrho_D(x, y)) < \delta = f(A(t))/2$, therefore from (2) it follows that $\neg(\varrho_D(x, y) \geqq A(t)/2)$. i.e.

$$d_t(x(t), y(t)) = (\varrho_D(x, y))\,(t) < (A(t))\,(t)/2 = \varepsilon\,.$$

Therefore $(x, y) \in V_t$. Then by (6) we have $\bigcap_{t \in F} W_t \subset \bigcap_{t \in F} V_t \subset U$ and therefore by (7) we get $U \in \mathcal{U}_f$.

Proposition 1. *Let $D = \{(M_t, d_t)\}_{t \in T}$ be a collection of metric spaces. Let $f \in \mathcal{M}(T)$ be a mapping continuous at the point Θ. Then $\mathcal{U}_D = \mathcal{U}_f$.*

Proof. By lemma 2 it suffices to prove that $\mathcal{U}_f \subset \mathcal{U}_D$. Let $U \in \mathcal{U}_f$. Then according to (4) there is a positive ε such that

$$(f \circ \varrho_D)^{-1}(\langle 0, \varepsilon)) \subset U\,. \tag{8}$$

Since f is continuous at the point Θ, we have

$$\exists F \subset T,\ F \neq \emptyset \textit{ finite } \exists \gamma > 0\ \forall y \in T^+:\quad (\forall t \in F\colon y(t) < \gamma) \Rightarrow f(y) < \varepsilon\,. \tag{9}$$

Denote $V = \bigcap_{t \in F} (\pi_t \times \pi_t)^{-1}(d_t^{-1}(\langle 0, \gamma)))$. Then $V \in \mathcal{U}_D$.

We show that $V \subset (f \circ \varrho_D)^{-1}(\langle 0, \varepsilon))$.
Let $(x, y) \in V$. Then $d_t(x(t), y(t)) < \gamma$ for all $t \in T$, therefore from (4) we get

$$(f \circ \varrho_D)\,(x, y) = f(\varrho_D(x, y)) < \varepsilon, \quad \text{i.e.} \quad (x, y) \in (f \circ \varrho_D)^{-1}(\langle 0, \varepsilon))\,.$$

Therefore $V \subset (f \circ \varrho_D)^{-1}(\langle 0, \varepsilon))$. Then from (8) we have $V \subset U$, therefore $U \in \mathcal{U}_D$.

Definition 5. *Let $D = \{(M_t, d_t)\}_{t \in T}$ be a collection of metric spaces. Define*

$$I_D = \{t \in T\colon \sup Im\, d_t < \infty\}\,, \tag{10}$$

$$S_D = \{t \in T\colon \forall \varepsilon > 0\colon d_t^{-1}((0, \varepsilon)) \neq \emptyset\}\,. \tag{11}$$

Theorem. *Let $D = \{(M_t, d_t)\}_{t \in T}$ be a collection of metric spaces. Let M_t be a nonempty set for each $t \in T$. Let $f \in \mathcal{M}(T)$. Then $\mathcal{U}_D = \mathcal{U}_f$ if and only if*

$$\forall \varepsilon > 0\ \exists F \subset T,\ F \neq \emptyset \textit{ finite } \exists \delta > 0\ \forall \alpha \in N^{(T-(I_D \cup F))}\ \exists a \in T^+:$$

$$\begin{aligned}
&(A) && \forall t \in T - (I_D \cup F)\colon a(t) \geqq \alpha(t),\\
&(B) && \forall t \in I_D - F\colon a(t) \geqq \sup Im\, d_t,\\
&(C) && \forall t \in F \cap S_D\colon a(t) \geqq \delta,\\
&(D) && f(a) < \varepsilon\,.
\end{aligned}$$

Proof. *Necessity.*
Let $\varepsilon > 0$. Since $\mathcal{U}_f \subset \mathcal{U}_D$, we have $(f \circ \varrho_D)^{-1}(\langle 0, \varepsilon/2)) \in \mathcal{U}_D$. Therefore according to (5) we have

$$\exists F \subset T,\ F \neq \emptyset \textit{ finite } \forall t \in F\ \exists U_t \in \mathcal{U}_t\colon \bigcap_{t \in F} (\pi_t \times \pi_t)^{-1}(U_t) \subset (f \circ \varrho_D)^{-1}(\langle 0, \varepsilon/2)).$$

Let $t \in F$. Since $U_t \in \mathscr{U}_t$, according to (4) there exists $\gamma_t > 0$ such that $d_t^{-1}(\langle 0, \gamma_t)) \subset U_t$. Denote $V = \bigcap_{t \in F} (\pi_t \times \pi_t)^{-1}(d_t^{-1}(\langle 0, \gamma_t)))$. Then obviously

$$(12) \qquad V \subset (f \circ \varrho_D)^{-1}(\langle 0, \varepsilon/2)).$$

Let $t \in F \cap S_D$. Then there are u_t, $v_t \in M_t$ such that

$$(13) \qquad 0 \langle d_t(u_t, v_t) < \gamma_t .$$

Put

$$(14) \qquad \delta = \min \{d_t(u_t, v_t) : t \in F \cap S_D\} > 0$$

(in case of $F \cap S_D = \emptyset$ let $\delta > 0$ be arbitrary).

Let $\alpha \in N^{(T-(I_D \cup F))}$. Let $t \in T - I_D$. Then there are p_t, $q_t \in M_t$ such that

$$(15) \qquad d_t(p_t, q_t) \geqq \alpha(t).$$

Denote $J = \{t \in I_D : \sup Im\, d_t > 0\}$.

Let $t \in J$. Then there are r_t, $s_t \in M_t$ such that

$$(16) \qquad d_t(r_t, s_t) > (1/2) \cdot \sup Im\, d_t .$$

Let $t \in T$. Since M_t is a nonempty set, choose an arbitrary element $w_t \in M_t$.

Define the mappings x, y: $T \rightarrow \bigcup_{t \in T} M_t$ as follows:

$$x(t) = \begin{matrix} u_t \\ p_t \\ r_t \\ w_t \end{matrix} \qquad y(t) = \begin{matrix} v_t \\ q_t \\ s_t \\ w_t \end{matrix} \qquad \begin{matrix} \text{for } t \in F \cap S_D \\ \text{for } t \in T - (I_D \cup F) \\ \text{for } t \in J - F \\ \text{for } t \in [I_D - (J \cup F)] \cup (F - S_D). \end{matrix}$$

Denote $a = 2 \cdot \varrho_D(x, y)$.

Now we show that a satisfies the conditions (A), (B), (C), (D). "A": Let $t \in T - (I_D \cup F)$. Then according to (15) we have

$$a(t) = (2 \cdot \varrho_D(x, y))\,(t) = 2d_t(x(t), y(t)) = 2d_t(p_t, q_t) \geqq \alpha(t).$$

"B": Let $t \in I_D - (J \cup F)$. Then we obtain

$$a(t) = 2d_t(x(t), y(t)) = 2d_t(w_t, w_t) = 0 = \sup Im\, d_t .$$

Let $t \in J - F$. Then from (16) we have

$$a(t) = 2d_t(x(t), y(t)) = 2d_t(r_t, s_t) > 2 \cdot (1/2) \cdot \sup Im\, d_t = \sup Im\, d_t .$$

Therefore $a(t) \geqq \sup Im\, d_t$ for all $t \in I_D - F$.

"C": Let $t\in F\cap S_D$. Then from (14) we get

$$a(t)=2d_t(x(t),\ y(t))=2d_t(u_t,\ v_t)\geqq\delta\,.$$

"D": Let $t\in F\cap S_D$. Then according to (13) we have

$$d_t(x(t),\ y(t))=d_t(u_t,\ v_t)<\gamma_t\,.$$

Let $t\in F-S_D$. Then

$$d_t(x(t),\ y(t))=d_t(w_t,\ w_t)=0<\gamma_t\,.$$

Therefore $d_t(x(t),\ y(t))<\gamma_t$ for each $t\in F$, i.e. $(x,\ y)\in V$. Then from (2) and (13) we obtain

$$f(a)\leqq 2f(\varrho_D(x,\ y))=2(f\circ\varrho_D)\,(x,\ y)<2\cdot\varepsilon/2=\varepsilon\,.$$

Sufficiency. By lemma 2 it suffices to prove that $\mathcal{U}_f\subset\mathcal{U}_D$. Let $U\in\mathcal{U}_f$. Then according to (4) there is a positive ε such that

(17) $$(f\circ\varrho_D)^{-1}(\langle 0,2\varepsilon))\subset U\,.$$

Then by the hypotheses we have

$$\exists F\subset T,\ F\neq\emptyset\ \textit{finite}\ \exists\delta>0\ \forall\alpha\in N^{(T-(I_D\cup F))}\ \exists a\in T^+:(A)-(D).$$

Let $t\in F-S_D$. Then there is $\gamma_t>0$ such that $d_t^{-1}((0,\ \gamma_t)=\emptyset$. Denote $\gamma=$ $=\min\{\gamma_t:\ t\in F-S_D\}>0$, in case of $F-S_D=\emptyset$ let $\gamma>0$ be arbitrary. Then

(18) $$d_t^{-1}((0,\ \gamma))=\emptyset\ \text{ for each }\ t\in F-S_D\,.$$

Denote $A=\bigcap\limits_{t\in F}(\pi_t\times\pi_t)^{-1}(d_t^{-1}(\langle 0,\ \min\{\gamma,\ \delta\}))$. Then $A\in\mathcal{U}_D$.

Let $(x,\ y)\in A$. Then

(19) $$d_t(x(t),\ y(t))<\min\{\gamma,\ \delta\}\ \text{ for each }\ t\in F.$$

Let $t\in T$. Then there is a positive integer n_t such that

$$d_t(x(t),\ y(t))\leqq n_t\,.$$

Define a mapping α: $(T-(I_D\cup F))\to N$ by

$$\alpha(t)=n_t\,.$$

Then by the hypothesis there is $a\in T^+$ satisfying $(A)-(D)$. We show that $\varrho_D(x,\ y)\leqq a$.

Let $t\in I_D-F$. Then from (8) we have

$$d_t(x(t),\ y(t))\leqq\sup\ Im\ d_t\leqq a(t)\,.$$

Let $t \in F \cap S_D$. Then from (19) and (C) we obtain

$$d_t(x(t), y(t)) \leqq \delta \leqq a(t).$$

Let $t \in T - (I_D \cup F)$. Then from (A) we get

$$d_t(x(t), y(t)) \leqq \alpha(t) \leqq a(t).$$

Let $t \in F - S_D$. Then from (19) and (18) we have

$$d_t(x(t), y(t)) = 0 \leqq a(t).$$

Therefore $(\varrho_D(x, y))(t) = d_t(x(t), y(t)) \leqq a(t)$ for each $t \in T$, i.e. $\varrho_D(x, y) \leqq a$. Then according to (2) and (D) we obtain

$$(f \circ \varrho_D)(x, y) = f(\varrho_D(x, y)) \leqq 2 \cdot f(a) < 2\varepsilon,$$

therefore $(x, y) \in (f \circ \varrho_D)^{-1}(\langle 0, 2\varepsilon))$, i.e. $A \subset (f \circ \varrho_D)^{-1}(\langle 0, 2\varepsilon))$. Then according to (17) we obtain $A \subset U$, therefore $U \in \mathscr{U}_D$.

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Matematický ústav SAV
Komenského 14
041 54 Košice

966 54 Tekovské Nemce 261
okr. Žiar nad Hronom

О МЕТРИЗАЦИИ РАВНОМЕРНОЙ СТРУКТУРЫ ПРОИЗВЕДЕНИЯ МЕТРИЧЕСКИХ ПРОСТРАНСТВ

Ян Борсик—Йозеф Добош

Резюме

Пусть T — некоторое непустое множество. Обозначим T^+ множество всех неотрицательных вещественных функций, определенных на множестве T. Пусть $f: T^+ \to R$ — функция, для которой

$$d(x, y) = f(d_t(x(t), y(t)))$$

является метрикой на множестве $\prod_{t \in T} M_t$ для каждого семейства метрических пространств (M_t, d_t) $(t \in T)$. В настоящей работе мы предлагаем необходимое и достаточное условие метризации равномерной структуры произведения при помощи метрики d.