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ON A CERTAIN LATTICE OF TOPOLOGIES ON A PRODUCT OF METRIC SPACES

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Introduction

Let T be a nonempty set. Denote by R the real line and by T^+ the set of all non-negative functions a: $T \to R$. Denote by $\mathcal{M}(T)$ the set of all functions f: $T^+ \to R$ such that

$$d(x, y) = f(\{d_t(x(t), y(t))\}_{t \in T}) \tag{1}$$

is a metric on the set $\prod_{t \in T} M_t$ for every collection of metric spaces $\{(M_t, d_t)\}_{t \in T}$.

In [1] we investigate the metrizability by the metric d of the product topology on $\prod_{t \in T} M_t$.

In the present paper we extend some results of [1]. In the special case of the set T being finite, the paper presents a complete characterization of the lattice of topologies on the set $\prod_{t \in T} M_t$ generated by the set $\mathcal{M}(T)$.

1. Preliminaries

1.1. Notation. If δ is a binary relation on R, define the binary relation δ_T on R^T as follows: $x\delta_T y$ if and only if $x(t)\delta y(t)$ for each $t \in T$. Define the function θ_T: $T \to R$ by $\theta_T(t) = 0$ for each $t \in T$. If $T = \{t\}$, we write δ_t and θ_t.

In paper [1] the following results (1.2—1.6) are proved.

1.2. Lemma. *Let $f \in \mathcal{M}(T)$. Then*

$$\forall a, b \in T^+: f(a + b) \leqq f(a) + f(b), \tag{2}$$

$$\forall a, b \in T^+: a \leqq_T 2b \Rightarrow f(a) \leqq 2f(b). \tag{3}$$

1.3. Theorem. *Let f: $T^+ \to R$. Then $f \in \mathcal{M}(T)$ if and only if*

$$\forall x \in T^+ : f(x) = 0 \Leftrightarrow x = \theta_T, \tag{4}$$

$$\forall x, y, z \in T^+ : (x \leqq_T y + z \,\&\, y \leqq_T x + z \,\&\, z \leqq_T x + y) \Rightarrow \Rightarrow f(x) \leqq f(y) + f(z). \tag{5}$$

1.4. Proposition. *Let the set T be finite. Let $f \in \mathcal{M}(T)$. Then f is continuous (we consider T^+ by subspace of the topological product R^T) if and only if*

$$\forall \varepsilon > 0 \; \exists x \in T^+, \; x >_T \theta_T : f(x) < \varepsilon.$$

1.5. Notation. *Let $S \subset T$ be a nonempty set. Define a mapping $i_{S,T}$: $S^+ \to T^+$* as follows

$$(i_{S,T}(a))(t) = \begin{cases} a(t) & \text{for} \quad t \in S, \\ 0 & \text{for} \quad t \in T - S, \end{cases}$$

for each $a \in S^+$. If $S = \{s\}$ we write $i_{s,T}$.

1.6. Lemma. *Let $f \in \mathcal{M}(T)$. Then $(f \circ i_{S,T}) \in \mathcal{M}(S)$.*

1.7. Remark. The mapping $i_{S,T}$ is continuous (see [2], p. 59, Theorem 1).

1.8. Proposition. *Let Q be a nonempty finite set. Let $f \in \mathcal{M}(Q)$. Then f is continuous if and only if $f \circ i_{q,Q}$ is continuous for each $q \in Q$.*

Proof. $\Rightarrow$: By 1.7.

$\Leftarrow$: Let $\varepsilon > 0$, $q \in Q$. Since by 1.6 we have

$$(f \circ i_{q,Q}) \in \mathcal{M}(\{q\}),$$

by 1.4 we obtain

$$\exists x_q \in \{q\}^+, \; x_q >_q \theta_q : (f \circ i_{q,Q})(x_q) < \varepsilon / (\text{card } Q).$$

We put

$$a = \sum_{q \in Q} i_{q,Q}(x_q).$$

Thus $a \in Q^+$, $a >_Q \theta_Q$ and by 1.2 we have

$$f(a) \leqq \sum_{q \in Q} (f \circ i_{q,Q})(x_q) < \varepsilon.$$

Then by 1.4 the function f is continuous.

1.9. Notation. For each $f \in \mathcal{M}(T)$ we put

$$F(f) = \{t \in T; \; f \circ i_{t,T} \text{ is continuous}\}.$$

Define a function j_T: $T^+ \to R$ as follows

$$j_T(x) = \begin{cases} 0 & \text{for} \quad x = \theta_T, \\ 1 & \text{for} \quad x \neq \theta_T. \end{cases}$$

The following Example shows that the condition "finite" in Proposition 1.8 cannot be omitted.

1.10. Example. Let P be a nonempty set. Define a mapping $f\colon P^+ \to R$ as follows

$$f(x) = \sup \{\min (1, x_t);\ t \in P\}.$$

Then $f \in \mathcal{M}(P)$, $F(f) = P$ and f is continuous if and only if P is finite.

1.11. Corollary. *Let the set T be finite. Let $f \in \mathcal{M}(T)$. Then $f \circ i_{S,T}$ is continuous if and only if $S \subset F(f)$.*

Proof. By 1.6 we have $(f \circ i_{S,T}) \in \mathcal{M}(S)$. Then by 1.8 we obtain that $f \circ i_{S,T}$ is continuous if and only if $f \circ i_{s,T} = (f \circ i_{S,T}) \circ i_{s,S}$ is continuous for each $s \in S$.

1.12. Proposition. *Let the set T be finite, $\emptyset \neq S \subset T$. Let $h \in \mathcal{M}(T)$ be continuous. Define a mapping $h_S\colon T^+ \to R$ as follows*

$$h_S(x) = \begin{cases} h(x)/(1 + h(x)) & \text{for } x \in \operatorname{Im}(i_{S,T}), \\ 1 & \text{otherwise.} \end{cases}$$

Then $h_S \in \mathcal{M}(T)$ and $F(h_S) = S$.

Proof. Let $x \in T^+$. Then $h_S(x) = 0 \Leftrightarrow h(x) = 0 \Leftrightarrow x = \theta_T$.

Let $x, y, z \in T^+$, $x \leqq_T y + z$, $y \leqq_T x + z$, $z \leqq_T x + y$. Since $h \in \mathcal{M}(T)$, by 1.3 we have

$$h(x) \leqq h(y) + h(z).$$

If $h_S(y) + h_S(z) < 1$, then $x, y, z \in \operatorname{Im}(i_{S,T})$, thus $h_S(x) = h(x)/(1 + h(x)) \leqq h(y)/(1 + h(y)) + h(z)/(1 + h(z)) = h_S(y) + h_S(z)$. If $h_S(y) + h_S(z) \geqq 1$, then $h_S(x) \leqq 1 \leqq h_S(y) + h_S(z)$. Then by 1.3 we have $h_S \in \mathcal{M}(T)$.

Since h is continuous, by 1.7 we obtain $h_S \circ i_{S,T} = (h/(1 + h)) \circ i_{S,T}$ is continuous. Thus by 1.11 we get $S \subset F(h_S)$.

Let $t \in T - S$. Since $h_S \circ i_{t,T} = j_T \circ i_{t,T}$ is not continuous, by 1.11 we have $t \in T - F(h_S)$. Thus $T - S \subset T - F(h_S)$.

2. Lattice of topologies generated by the set $\mathcal{M}(T)$

2.1. Notation. Let T be a nonempty set. Let $\{(M_t, d_t)\}_{t \in T}$ be a collection of metric spaces. We put $M = \prod_{t \in T} M_t$. For each $f \in \mathcal{M}(T)$ denote by $\mathcal{T}_f$ the topology on the set M derived from the metric (1). We put

$$\mathcal{L} = \{\mathcal{T}_f\colon f \in \mathcal{M}(T)\},$$
$$H = \{t \in T\colon M'_t \neq \emptyset\}$$

(where A' is the derived set of A).

2.2. Proposition. *Let f, $g \in \mathcal{M}(T)$. Let $\mathcal{T}_f \subset \mathcal{T}_g$. Then*

$$F(f) \supset F(g) \cap H.$$

Proof. Let $t \in F(g) \cap H$. Let $\varepsilon > 0$. Select $a \in M$ such that $a(t) \in M'_t$. Since $\mathcal{T}_f \subset \mathcal{T}_g$, there exists $\delta > 0$ such that

$$S_g(a, 2\delta) \subset S_f(a, \varepsilon). \tag{7}$$

Since $g \circ i_{t,T}$ is continuous, by 1.4 we have

$$\exists y \in \{t\}^+, \ y >_t \theta_t : (g \circ i_{t,T})(y) < \delta.$$

Let $q \in M_t$ such that $0 < d_t(a(t), q) < y(t)$. Define a mapping $b: T \to \bigcup_{t \in T} M_t$ as follows

$$b(s) = \begin{cases} q & \text{for } s = t, \\ a(s) & \text{otherwise.} \end{cases}$$

Define a mapping $x: \{t\} \to R$ as follows

$$x(t) = d_t(a(t), b(t)).$$

Then obviously $x \in \{t\}^+$, $x >_t \theta_t$. Since $(g \circ i_{t,T}) \in \mathcal{M}(\{t\})$ and $x \leqq_t y$, by 1.2 (3) we obtain $g(\{d_t(a(t), b(t))\}_{t \in T}) = (g \circ i_{t,T})(x) \leqq 2 \cdot (g \circ i_{t,T})(y) < 2\delta$. Thus $b \in S_g(a, 2\delta)$. Then by (7) we have $(f \circ i_{t,T})(x) = f(\{d_t(a(t), b(t))\}_{t \in T}) < \varepsilon$. Hence by 1.4 the function $f \circ i_{t,T}$ is continuous.

In the following it will be proved that if T is finite, then the topologies of metrics which are generated by functions from $\mathcal{M}(T)$ are determined by subsets of the set of all indices t, so that d_t is not discrete.

2.3. Theorem. *Let the set T be finite. Let f, $g \in \mathcal{M}(T)$. Then $\mathcal{T}_f \subset \mathcal{T}_g$ if and only if $F(f) \supset F(g) \cap H$.*

Proof. $\Rightarrow$: By 2.2.

$\Leftarrow$: Let $a \in M$, $\varepsilon > 0$. We show that

$$\exists \delta > 0 : S_g(a, \delta) \subset S_f(a, \varepsilon).$$

Let $\gamma > 0$ such that

$$\forall t \in T - H \ \forall b \in T^+ : (d_t(a(t), b(t)) < \gamma) \Rightarrow a(t) = b(t). \tag{8}$$

Let $\eta > 0$ such that

$$\forall t \in T - F(g) \ \forall x \in \{t\}^+, \ x >_t \theta_t : (g \circ i_{t,T})(x) \geqq \eta. \tag{9}$$

Let $t \in F(f)$. Since $f \circ i_{t,T}$ is continuous, there exists $x_t \in \{t\}^+$, $x_t >_t \theta_t$ such that

$$(f \circ i_{t,T})(x_t) < \varepsilon / (2 \text{ card } T). \tag{10}$$

We put

$$\delta_t = g(i_{t,T}(x_t))/2.$$

For each $t \in T - F(f)$ we put $x_t = \theta_t$. For each $t \in T$ define a function y_t: $\{t\} \to R$ by $y_t(t) = \gamma$ and put

$$\gamma_t = g(i_{t,T}(y_t))/2.$$

We put $\delta = \min(\{\delta_t\colon t \in F(f)\} \cup \{\gamma_t\colon t \in T\} \cup \{\eta/2\})$. Let $b \in S_g(a, \delta)$, $t \in F(f)$. Since $2g(\{d_u(a(u), b(u))\}_{u \in T}) < 2\delta \leqq g(i_{t,T}(x_t))$, by 1.2 (3) we obtain

$$2d_t(a(t), b(t)) < x_t(t).$$

Let $t \in T - H$. Since $2g(\{d_u(a(u), b(u))\}_{u \in T}) \leqq 2\delta \leqq g(i_{t,T}(y_t))$, by 1.2 (3) we have $2d_t(a(t), b(t)) < \gamma$. Then by (8) we get $a(t) = b(t)$.

Let $t \in T - F(g)$. Define a function u: $\{t\} \to R$ by

$$u(t) = d_t(a(t), b(t)).$$

Since $i_{t,T}(u) \leqq_T 2 \cdot \{d_v(a(v), b(v))\}_{v \in T}$, by 1.2 (3) we have

$$(g \circ i_{t,T})(u) \leqq 2 \cdot g(\{d_v(a(v), b(v))\}_{v \in T}) < 2\delta \leqq \eta.$$

Thus by (9) we set $a(t) = b(t)$.

Then by 1.2 and (10) we obtain

$$f(\{d_t(a(t), b(t))\}_{t \in T}) \leqq$$

$$\leqq 2 \cdot f\left(\sum_{t \in T} i_{t,T}(x_t)\right) \leqq 2 \cdot \sum_{t \in T} (f \circ i_{t,T})(x_t) < \varepsilon.$$

Hence $b \in S_f(a, \varepsilon)$.

2.4. Corollary. *Let the set T be finite. Let $f, g \in \mathcal{M}(T)$. Then $\mathcal{T}_f = \mathcal{T}_g$ if and only if $H \cap F(f) = H \cap F(g)$.*

The following Example shows that the condition "finite" in Theorem 2.3 cannot be omitted.

2.5. Example. Let W be a infinite set. Let a: $W \to N$ be a surjection (when N denotes the set of all natural numbers). Define a mapping g: $W^+ \to R$ as follows

$$g(x) = \sup\{\min(1, a_t \cdot x_t);\ t \in W\}.$$

Then $g \in \mathcal{M}(W)$. Let $f \in \mathcal{M}(W)$ be the function from Example 1.10. Consider the collection of metric spaces $\{(M_t, d_t)\}_{t \in W}$ given by $M_t = R$, $d_t(x, y) = |x - y|$ for each $t \in W$. Evidently $S_g(\theta_W, 1) \in \mathcal{T}_g$. We prove that $S_g(\theta_W, 1) \notin \mathcal{T}_f$. Since for every constant function $u \in W^+$, $u \neq \theta_W$ we have $g(u) = 1$, for every $\varepsilon > 0$ we obtain

$$S_f(\theta_W, \varepsilon) \not\subset S_g(\theta_W, 1).$$

Thus $S_u(\theta_W, 1)$ is not the neighbourhood of θ_W in $\mathcal{T}_f$. Hence $S_u(\theta_W, 1) \notin \mathcal{T}_f$. Then $\mathcal{T}_u \not\subset \mathcal{T}_f$, but $F(f) = F(g) = W$.

2.6. Proposition. *Let the set T be finite. Let $h \in \mathcal{M}(T)$ be continuous. Put $h_0 = j_T$. Then*

$$\mathcal{S} = \{\mathcal{T}_{h_S} : S \subset H\}.$$

Proof. Let $f \in \mathcal{M}(T)$. We put $S = H \cap F(f)$. Then by 1.12 we have $H \cap F(h_S) = H \cap S = H \cap F(f)$. Hence by 2.3 we obtain

$$\mathcal{T}_f = \mathcal{T}_{h_S}.$$

2.7. Remark. It is not difficult to prove that the partially ordered set $(\mathcal{L}, \subset)$ is a lattice.

2.8. Theorem. *The lattice $(\mathcal{L}, \subset)$ is dually isomorphic to the lattice* $(\exp H, \subset)$.

Proof. Define a mapping Ω: $\mathcal{L} \to \exp H$ by

$$\Omega(\mathcal{T}_f) = H \cap F(f)$$

for each $f \in \mathcal{M}(T)$. By 1.12, 2.3, 2.4 and 2.6 the mapping Ω is a dual isomoprhism.

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ОБ ОДНОЙ СТРУКТУРЕ ТОПОЛОГИЙ НА ПРОИЗВЕДЕНИИ МЕТРИЧЕСКИХ ПРОСТРАНСТВ

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Резюме

Пусть T является непустым конечным множеством. Обозначим T^* множество всех неотрицательных вещественных функций, определенных на множестве T. Обозначим $\mathcal{M}(T)$ множество всех функций f: $T^* \to R$, для которых

$$d(x, y) = f(\{d_t(x(t), y(t))\}_{t \in T})$$

является метрикой на множестве

$$\prod_{t \in T} M_t$$

для каждого семейства метрических пространств $\{(M_t, d_t)\}_{t \in T}$. В настоящей работе мы предлагаем характеризацию структуры топологий, порожденной множеством $\mathcal{M}(T)$.