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A NOTE ON THE FUNCTIONS THE GRAPHS OF WHICH ARE CLOSED SETS

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Let (X, ϱ), (Y, σ) be two metric spaces. Denote by $U(X, Y)$ the set of all functions f: $X \to Y$, the graphs of which are closed subsets of the space $(X \times Y, \tau)$ where $\tau = \sqrt{(\varrho^2 + \sigma^2)}$. Denote $G(f)$ the graph of the function f: $X \to Y$.

In [1] the following theorem is proved:

Let $f_n \in U(X, Y)$ and let $\{f_n\}_{n=1}^{\infty}$ be almost uniformly convergent to f (i.e. $\{f_n\}_{n=1}^{\infty}$ is uniformly convergent to f on each compact $K \subset X$). Then $f \in U(X, Y)$.

We shall show that the above mentioned assertion is not true. Further a condition under which the set $U(X, Y)$ is closed with respect to quasi-uniform convergence will be given.

Let R, Q and N be sets of all real, rational and positive integer numbers, respectively.

Example. Denote $X = \{1/m;\ m \in N\} \cup \{0\}$, $Y = Q$. Let $\{a_n\}_{n=1}^{\infty}$ be an increasing sequence of points $a_n \in R - Q$, such that $a_n \to 1$. Let $n \in N$. Let $\{b_{nk}\}_{k=1}^{\infty}$ be a sequence of points $b_{nk} \in Q \cap (a_n, a_{n+1})$, such that $b_{nk} \to a_{n+1}$. Define a function f_n: $X \to Y$ by $f_n(0) = 2$, $f_n(1/m) = b_{nm}$ for every $m \in N$. Define a function f: $X \to Y$ by $f(0) = 2$, $f(1/m) = 1$ for every $m \in N$. We show that $f_n \in U(X, Y)$ $(n = 1, 2, \ldots)$, $f_n \rightrightarrows f$, but $f \notin U(X, Y)$.

Let $n \in N$. We show that $f_n \in U(X, Y)$. Let $c_p \in G(f_n)$ $(p = 1, 2, \ldots)$, $c_p \to c_0 \in X \times Y$. We show that

$$\exists p \in N\ \forall q \in N, \quad q \geqq p:\ c_q = c_p$$

(then obviously $c_0 \in G(f_n)$). Assume the contrary. Then sequences $\{c_p\}_{p=1}^{\infty}$, $\{(1/m, f_n(1/m))\}_{m=1}^{\infty}$ have a common subsequence $\{d_q\}_{q=1}^{\infty}$ and $d_q \to c_0$. Since $\{(1/m, f_n(1/m))\}_{m=1}^{\infty}$ is a Cauchy sequence, $(1/m, f_n(1/m)) \to c_0$, a contradiction. We now show that $f_n \rightrightarrows f$. Let $\varepsilon > 0$. Since $a_n \to 1$, we have

$$\exists n_0 \in N\ \forall n \geqq n_0:\ |a_n - 1| < \varepsilon$$

Let $m \in N$. Then obviously

$$\forall n \geqq n_0: |f_n(1/m) - f(1/m)| = |b_{nm} - 1| < |a_n - 1| < \varepsilon,$$
$$|f_n(0) - f(0)| = |2 - 2| < \varepsilon$$

Hence

$$\forall n \geqq n_0 \ \forall x \in X: |f_n(x) - f(x)| < \varepsilon$$

i.e. $f_n \rightrightarrows f$. If $c_p = (1/p, f(1/p)) = (1/p, 1)$, $p \in N$, then obviously $c_p \in G(f)$, $c_p \to (0, 1) \notin G(f)$. Consequently $f \notin U(X, Y)$.

In the next text we shall give a sufficient condition for the closedness of $U(X, Y)$ with respect to quasi-uniform convergence. Recall the definition of quasi-uniform convergence.

Definition. Let X be a set and Y a metric space (with the metric d). Let $f_n: X \to Y$ ($n = 1, 2, \ldots$). A sequence $\{f_n\}_{n=1}^{\infty}$ is said to converge quasi-uniformly to $f: X \to Y$ if

(i) $\forall x \in X: f_n(x) \to f(x)$,

(ii) $\forall \varepsilon > 0 \ \forall m \in \{0, 1, 2, \ldots\} \ \exists p \in N \ \forall x \in X$:

$$\min\{d(f_{m+1}(x), f(x)), \ldots, d(f_{m+p}(x), f(x))\} < \varepsilon \text{ (see [2], p. 143).}$$

Theorem. Let (X, ϱ) be a metric space. Let (Y, σ) be a metric space, in which every closed and bounded set is compact. Let $f_n \in U(X, Y)$ and let $\{f_n\}_{n=1}^{\infty}$ be quasi-uniformly convergent to f. Then $f \in U(X, Y)$.

Proof. Let $(x_p, y_p) \in G(f)$ ($p = 1, 2, \ldots$), $(x_p, y_p) \to (x_0, y_0)$. We show that each neighbourhood of the point $(x_0, f(x_0))$ contains the point (x_0, y_0) (then obviously $(x_0, y_0) \in G(f)$). Let U be a neighbourhood of the point $(x_0, f(x_0))$. Then

$$\exists h \in N: S((x_0, f(x_0)), 2/h) \subset U \tag{1}$$

Since $f_n(x_0) \to f(x_0)$, we have

$$\exists m > h \ \forall k > m: \sigma(f_k(x_0), f(x_0)) < 1/h$$

Therefore

$$\forall k > m: \tau((x_0, f(x_0)), (x_0, f_k(x_0))) < 1/h \tag{2}$$

Let $k \in N$. Since $\{f_n\}_{n=1}^{\infty}$ is quasi-uniformly convergent to f, we have $\exists q \in N$ $\forall x \in X$:

$$\min\{\sigma(f_{m+1}(x), f(x)), \ldots, \sigma(f_{m+q}(x), f(x))\} < 1/(h+1)$$

Denote $A_i = \{p \in N: \sigma(f_{m+i}(x_p), f(x_p)) < 1/(h+1)\}$ for each $i \in \{1, \ldots, q\}$. Then obviously

$$N = \bigcup_{i=1}^{q} A_i$$

Hence there exist $j \in \{1, \ldots, q\}$, such that the set A_j is infinite. Let $\{x_{p_r}\}_{r=1}^{\infty}$ be a subsequence of the sequence $\{x_p\}_{p=1}^{\infty}$, such that

$$\forall r \in N: p_r \in A_j$$

Denote $z_r = x_{p_r}$ for each $r \in N$. Then

$$\forall r \in N: \sigma(f_{m+j}(z_r), f(z_r)) < 1/(h+1) \tag{3}$$

Since $f(z_r) \to y_0$ the set $\{f(z_r): r \in N\}$ is bounded. Denote $B = S(y_0, \sup\{\sigma(f(z_r), y_0): r \in N\} + 1/h)$. Then

$$\forall r \in N: f_{m+j}(z_r) \in B$$

Since $\bar{B}$ is compact (in Y) there exists a subsequence $\{t_s\}_{s=1}^{\infty}$ of the sequence $\{z_r\}_{r=1}^{\infty}$, such that $f_{m+j}(t_s) \to f_{m+j}(x_0)$. According to (3) we have

$$\forall s \in N: \tau((t_s, f(t_s)), (t_s, f_{m+j}(t_s))) < 1/(h+1)$$

Since $f_{m+j} \in U(X, Y)$, we obtain $(s \to \infty)$

$$\tau((x_0, y_0), (x_0, f_{m+j}(x_0))) \leqq 1/(h+1) < 1/h \tag{4}$$

It follows from (2) and (4) that $\tau((x_0, y_0), (x_0, f(x_0))) \leqq \tau((x_0, y_0), (x_0, f_{m+j}(x_0))) + \tau((x_0, f(x_0)), (x_0, f_{m+j}(x_0))) < 2/h$, i.e.

$$(x_0, y_0) \in S((x_0, f(x_0)), 2/h)$$

According to (1) we obtain $(x_0, y_0) \in U$. The proof is complete.

Note that the above theorem remains valid if the assumption "$\{f_n\}_{n=1}^{\infty}$ is quasi-uniformly convergent to f" is replaced by "$\{f_n\}_{n=1}^{\infty}$ is almost uniformly convergent to f". For this case the proof is analogical.

BIBLIOGRAPHY

[1] Костырко, П.–Нйбрун, Т.–Шалат, Т.: О функциях, графы которых являются замкнутыми множествами. Acta F.R.N. Univ. Comen. 10, 3, 1965, 51—61.

[2] Sikorski, R.: Funkcje rzeczywiste I. PWN, Warszawa, 1958.

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SÚHRN

POZNÁMKA O FUNKCIÁCH, KTORÝCH GRAFY SÚ UZAVRETÉ MNOŽINY

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Nech X, Y sú dva metrické priestory. Označme $U(X, Y)$ množinu všetkých funkcií f: $X \to Y$, ktorých grafy sú uzavreté množiny. V práci sa udáva postačujúca podmienka, pri ktorej je množina $U(X, Y)$ uzavretá vzhľadom na kvázirovnomernú konvergenciu.

РЕЗЮМЕ

ЗАМЕЧАНИЕ О ФУНКЦИЯХ, ГРАФЫ КОТОРЫХ ЯВЛЯЮТСЯ ЗАМКНУТЫМИ МНОЖЕСТВАМИ

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Пусть X, Y два метрические пространства. Обозначим знаком $U(X, Y)$ систему всех тех отображений f: $X \to Y$, графы которых являются замкнутыми множествами. В работе данное достаточное условие, при котором система $U(X, Y)$ является замкнутой относительно квазиравномерной сходимости.