UNIVERSITAS COMENIANA
ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
XL—XLI — 1982

CLIQUISH FUNCTIONS, RIEMANN INTEGRABLE FUNCTIONS AND QUASI-UNIFORM CONVERGENCE

JOZEF DOBOŠ, TIBOR ŠALÁT, Bratislava

Introduction

In this paper we shall prove some results on cliquish functions and on the quasi-uniform convergence of sequences of cliquish functions and Riemann integrable functions, respectively.

Let X be a topological and Y a metric spaces (with the metric d). The function $f: X \to Y$ is said to be cliquish at the point $p \in X$ if for each neighbourhood $U(p)$ of the point p and each $\varepsilon > 0$ there exists an open set $U \neq \emptyset$, $U \subset U(p)$ such that $d(f(x), f(y)) < \varepsilon$ holds for each two points $x, y \in U$. The function f is said to be cliquish on X if it is cliquish at each point $x \in X$ (cf. [10]).

Let X be a set and Y a metric space (with the metric d). The sequence $\{f_n\}_{n=1}^{\infty}$ of functions $f_n: X \to Y$ $(n = 1, 2, \ldots)$ is said to be quasi-uniformly convergent to the limit function $f: X \to Y$ if for each $x \in X$ we have $\lim_{n \to \infty} f_n(x) = f(x)$ and if for each $\varepsilon > 0$ and each $m \in \{0, 1, 2, \ldots\}$ there exists such a positive integer r that for each $x \in X$ the inequality

$$\min \{d(f_{m+1}(x), f(x)), \ldots, d(f_{m+r}(x), f(x))\} < \varepsilon$$

holds (cf. [8], p. 143).

Let X, Y be two topological spaces, let $f: X \to Y$. Denote by $C_f(A_f)$ the set of all such points at which the function f is continuous (cliquish). We put $D_f = X - C_f$.

The topological space X is said to be a Baire space if every non-empty open subset of X is a set of the second Baire category in X (cf. [2]).

1. Cliquish functions and pointwise discontinuous functions

We shall give a characterization of cliquish functions f defined on a Baire space using the sets D_f.

Theorem 1.1. Let X be a Baire and Y a metric spaces. The function f: $X \to Y$ is cliquish on X if and only if the set D_f is a set of the first Baire category.

Proof. If f is cliquish on X, then D_f is a set of the first Baire category in X (cf. [7]).

Conversely, let D_f be a set of the first category in X. Since $X - A_f \subset D_f$, the set $X - A_f$ is a set of the first category in X, too. It is well-known (cf. [6]) that the set $X - A_f$ is open. Since X is a Baire space, the set $X - A_f$ is empty, i.e. $X = A_f$.

Remark. In Theorem 1.1 the condition "X is a Baire space" cannot be omitted. Let X be the space of all rational numbers and R the space of all real numbers (with Euclidean metrics). Then there exists a function f: $X \to Y$ such that $A_f = \emptyset$ (cf. [6], Theorem 2). Thus f is not cliquish and D_f is a set of the first category in X.

The following theorem generalizes a certain result from [3], p. 38.

Theorem 1.2. Let X be a Baire and Y a metric spaces. The function f: $X \to Y$ is pointwise discontinuous (i.e. C_f is dense in X) if and only if the set D_f is a set of the first Baire category in X.

Proof. Let D_f be a set of the first category in X. If G is a non-empty open set in X, then G is a set of the second category in X. Thus $G - D_f$ is non-empty, hence $G \cap C_f \neq \emptyset$. The density of C_f follows.

Let f be pointwise discontinuous. It is well-known that C_f is a G_δ-set in X (cf. [4], p. 78). Thus C_f is residual in X (cf. [5], p. 49), hence D_f is a set of the first category in X.

Remark. a) In Theorem 1.2 the condition "X is a Baire space" cannot be omitted. Let X be the space of all rational numbers of the interval $\langle 0, 1 \rangle$ and $Y = R$ (with the Euclidean metrics). Let f: $X \to Y$, $f\left(\frac{p}{q}\right) = \frac{1}{q}$ for $\frac{p}{q} \in X$, $(p, q) = 1$, $q > 0$. Then D_f is a set of the first category in X, but f is not pointwise discontinuous.

b) In Theorem 1.2 the condition "Y is a metric space" cannot be replaced by condition "Y is a topological space". Let $X = \{x_1, x_2, x_3\}$, $T = \{\emptyset, \{x_1, x_2\}, X\}$, $Y = \{a, b\}$, $S = \{\emptyset, \{b\}, Y\}$, where $x_i \neq x_j$ for $i \neq j$ and $a \neq b$. Then (X, T) is a Baire space and (Y, S) is a topological space. Let f: $X \to Y$, $f(x_1) = f(x_3) = a$, $f(x_2) = b$. Then f is pointwise discontinuous, but D_f is a set of the second category in X.

Combining Theorem 1.1 with Theorem 1.2 we obtain at once the following result which is an extension of a certain result of the paper [7] (Theorem (iv')).

Theorem 1.3. Let X be a Baire and Y a metric spaces. The function f: $X \to Y$

is cliquish on X if and only if it is pointwise discontinuous on X.

2. Quasi-uniform convergence and the classes of cliquish and Riemann integrable functions

In this part of the paper we shall study limit functions of quasi-uniformly convergent sequences of cliquish functions and Riemann integrable functions, respectively.

We now extend a certain result of the paper [1].

Theorem 2.1. Let X be a Baire and Y a metric spaces. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of cliquish functions f_n: $X \to Y$ $(n=1, 2, \ldots)$ which converges quasi-uniformly to the function f: $X \to Y$. Then f is a cliquish function on X.

Proof. From quasi-uniform convergence we get at once

$$\bigcap_{n=1}^{\infty} C_{f_n} \subset C_f, \quad \text{i.e.} \quad D_f \subset \bigcup_{n=1}^{\infty} D_{f_n}$$

(cf. [8], p. 143 and [9], p. 167). Since D_{f_n} $(n=1, 2, \ldots)$ is a set of the first category in X, D_f is a set of the first category in X, too. On account of Theorem 1.1 we can conclude that f is cliquish on X.

It is well-known that if a sequence $\{f_n\}_{n=1}^{\infty}$ of Riemann integrable functions on $\langle a, b\rangle$ converges uniformly to f on $\langle a, b\rangle$, then the function f is again Riemann integrable on $\langle a, b\rangle$ and

$$\int_a^b f(t)\,\mathrm{d}t = \lim_{n\to\infty} \int_a^b f_n(t)\,\mathrm{d}t \tag{1}$$

In connection with this fact we shall prove the following theorem.

Theorem 2.2. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of Riemann integrable function f_n: $\langle a, b\rangle \to R$ $(n=1, 2, \ldots)$ which converges quasi-uniformly to f: $\langle a, b\rangle \to R$. Then f is a Riemann integrable function on $\langle a, b\rangle$.

Proof. At first we shall prove that f is bounded. It follows from the quasi-uniform convergence of the sequence $\{f_n\}_{n=1}^{\infty}$ that there exists such a positive integer r that for each $x \in \langle a, b\rangle$ the inequality

$$\min\{|f_1(x)-f(x)|, \ldots, |f_r(x)-f(x)|\} < 1 \tag{2}$$

holds. Since f_j $(j=1, 2, \ldots, r)$ are bounded, there exists such a positive number K that for each $x \in \langle a, b\rangle$ we have

$$|f_j(x)| \leqq K \quad (j=1, 2, \ldots, r) \tag{3}$$

From (2) and (3) we get $|f(x)| \leqq K+1$ for each $x \in \langle a, b\rangle$. Thus f is bounded on $\langle a, b\rangle$.

It suffices to show that the Lebesgue measure of the set D_f is zero (cf. [8], p. 489). Since the Lebesgue measure of each of the sets D_{f_n} $(n=1, 2, \ldots)$ is zero (cf. [8], p. 489) and as we already have seen $D_f \subset \bigcup_{n=1}^{\infty} D_{f_n}$, it is clear that the Lebesgue measure of the set D_f is zero. This ends the proof.

Remark. There exists a sequence $\{f_n\}_{n=1}^{\infty}$ of continuous functions $f_n: \langle 0, 1\rangle \to R$ $(n=1, 2, \ldots)$ quasiuniformly converging to a function $f: \langle 0, 1\rangle \to R$ such that (1) does not hold. We put $f_n(x)=0$ for $0 \leqq x \leqq \frac{1}{n+1}$ and for $\frac{1}{n} \leqq x \leqq 1$. Further we put $f_n(x_n)=2n(n+1)$

$$\left(x_n = \frac{1}{2}\left(\frac{1}{n}+\frac{1}{n+1}\right)\right) \quad (n=1, 2, \ldots)$$

Let f_n be linear and continuous on the intervals $\left\langle \frac{1}{n+1}, x_n\right\rangle, \left\langle x_n, \frac{1}{n}\right\rangle$ $(n=1, 2, \ldots)$. Evidently the functions f_n $(n=1, 2, \ldots)$ are continuous on $\langle 0, 1\rangle$ and the sequence $\{f_n\}_{n=1}^{\infty}$ converges quasi-uniformly to the function $f: \langle 0, 1\rangle \to R$, $f(x)=0$ for each $x \in \langle 0, 1\rangle$.

Since $\int_0^1 f(t)\,\mathrm{d}t = 0$ and $\int_0^1 f_n(t)\,\mathrm{d}t = 1$ $(n=1, 2, \ldots)$, (1) is not true.

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Author's address:

Jozef Doboš, Tibor Šalát
MFF UK, Katedra algebry a teórie čísiel,
Matematický pavilón
Mlynská dolina
Bratislava
842 15

Received: 13. 5. 1980

SÚHRN

KĽUKATÉ FUNKCIE, FUNKCIE INTEGROVATEĽNÉ V RIEMANNOVOM ZMYSLE A KVÁZIROVNOMERNÁ KONVERGENCIA

J. Doboš, T. Šalát, Bratislava

Nech X je Baireho topologický priestor a Y je metrický priestor. Nech $\{f_n\}_{n=1}^{\infty}$ je postupnosť kľukatých funkcií (cliquish functions), f_n: $X \to Y$ ($n = 1, 2, \ldots$), ktorá kvázirovnomerne konverguje k funkcii f: $X \to Y$. Potom f je zase kľukatá funkcia.

Nech $a, b \in R$, $a < b$, nech $\{f_n\}_{n=1}^{\infty}$ je postupnosť reálnych integrovateľných funkcií v Riemannovom zmysle na intervale $\langle a, b \rangle$. Ak $\{f_n\}_{n=1}^{\infty}$ konverguje kvázirovnomerne k funkcii f: $\langle a, b \rangle \to R$, tak f je integrovateľná funkcia na intervale $\langle a, b \rangle$ v Riemannovom zmysle.

РЕЗЮМЕ

ИЗВИЛИСТЫЕ ФУНКЦИИ, ФУНКЦИИ ИНТЕГРЫРУЕМЫЕ В СМЫСЛЕ РИМАНА И КВАЗИРАВНОМЕРНАЯ СХОДИМОСТЬ

Я. Добош, Т. Шалат, Братислава

Пусть X топологическое пространство Бера и $У$ метрическое пространство. Пусть

$$\{f_n\}_{n=1}^{\infty}$$

последовательность извилистых функций (cliquish functions) f_n: $X \to Y$ ($n = 1, 2, \ldots$), которая сходится квазиравномерно к функции f: $X \to Y$. Потом f извилистая функция.

Пусть $a, b \in R$, $a < b$, пусть

$$\{f_n\}_{n=1}^{\infty}$$

последовательность функций f_n: $\langle a, b \rangle \to R$ ($n = 1, 2, \ldots$) интегрыруемых в смысле Римана на $\langle a, b \rangle$. Эсли

$$\{f_n\}_{n=1}^{\infty}$$

сходится квазиравномерно к функции f на $\langle a, b \rangle$, то f тоже интегрыруема на $\langle a, b \rangle$ функция в смысле Римана.