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SOME GENERALIZATIONS OF THE NOTION OF CONTINUITY AND QUASI-UNIFORM CONVERGENCE

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It is well known that the sets of quasi-continuous, somewhat continuous and cliquish functions are all closed with respect to the uniform convergence (see [3], [5], [9]). The aim of this paper is to investigate whether or not those sets are closed with respect to the quasi-uniform convergence (see [7], p. 143).

Let X, Y be two topological spaces. A function $f : X \to Y$ is said to be quasi-continuous at a point $x_0 \in X$ if for each neighbourhood $U(x_0)$ of the point x_0 (in X) and each neighbourhood $V(f(x_0))$ of the point $f(x_0)$ (in Y) there exists an open set $U \subset U(x_0)$, $U \neq \emptyset$ such that $f(U) \subset V(f(x_0))$ (see [5]).

A function $f : X \to Y$ is said to be somewhat continuous if for each set $V \subset Y$ open in Y such that $f^{-1}(V) \neq \emptyset$ there exists an open set $U \subset X$, $U \neq \emptyset$ such that $U \subset f^{-1}(V)$ (see [3]).

Let X be a topological and Y a metric space (with the metric d). A function $f : X \to Y$ is said to be cliquish at a point $x_0 \in X$ if for each neighbourhood $U(x_0)$ of the point x_0 and each $\varepsilon > 0$ there exists an open set $U \subset U(x_0)$, $U \neq \emptyset$ such that $d(f(x'), f(x'')) < \varepsilon$ holds for every two points $x', x'' \in U$ (see [9]).

A function f defined on a topological space X is said to be quasi-continuous or cliquish on X if it is quasi-continuous or cliquish, respectively, at each point $x \in X$.

The property of the quasi-continuity is equivalent to the property of the semi-continuity (see [4], [6]).

Every function $f : X \to Y$ quasi-continuous on X is also somewhat continuous on X (see [8]).

Proposition 1. *There exists a sequence of functions $f_n : R \to R$ quasi-uniformly converging to $f : R \to R$ such that f_n is quasi-continuous but f is not somewhat continuous.*

Proof. Let the sequence $\{f_n\}_{n=1}^{\infty}$ of functions $f_n : R \to R$ be defined by

$$f_n(x) = \chi_{\langle 0,1/n \rangle}((-1)^n . x)$$

for all $x \in R$. Obviously $f_n \to f = \chi_{\{0\}}$. Let $\varepsilon > 0$, $m \in \{0, 1, 2, \ldots\}$. Denote $p = m + 2$. Then for $x \geqq 0$ we have $|f_{m+p-1}(x) - f(x)| \approx 0$, and for $x < 0$ we have

$|f_{m+p}(x) - f(x)| = 0$, i.e. $\forall x \in R : \min\{|f_{m+1}(x) - f(x)|, \ldots, |f_{m+p}(x) - f(x)|\} < \varepsilon$. Hence $\{f_n\}_{n=1}^{\infty}$ quasi-uniformly converges to f. We now show that f_n $(n = 1, 2, \ldots)$ are quasi-continuous functions. Let $n \in N$. Let U be an open neighbourhood of the point $x_0 = 0$ and V an open neighbourhood of the point $f_n(x_0)$. Then there exists $0 < \delta < 1/(2n)$ such that $(x_0 - 2\delta,\ x_0 + 2\delta) \subset U$. Denote $a = (-1)^n . \delta$ and $U_0 = (a - \delta,\ a + \delta)$. Then U_0 is open, $\emptyset \neq U_0 \subset U$ and $f_n(U_0) \subset V$. Hence f_n is quasi-continuous at the point $x_0 = 0$. Let U be an open neighbourhood of the point $x_1 = (-1)^n/n$ and V an open neighbourhood of the point $f_n(x_1)$. Then there exists $0 < \delta < 1/(2n)$ such that $(x_1 - 2\delta,\ x_1 + 2\delta) \subset U$. Denote $b = (-1)^n . (1/n - \delta)$ and $U_1 = (b - \delta,\ b + \delta)$. Then U_1 is open, $\emptyset \neq U_1 \subset U$ and $f_n(U_1) \subset V$. Hence f_n is quasi-continuous at the point $x_1 = (-1)^n/n$. Since f_n is continuous at each point $x \in R - \{0, (-1)^n/n\}$, conclude that f_n is quasi-continuous. Since $\operatorname{int} f^{-1}((1/2, 2)) = \operatorname{int}\{0\} = \emptyset$, f is not somewhat continuous.

Proposition 2. *There exists a nonempty set $M \subset R$ and a sequence of functions $f_n : M \to R$ quasi-uniformly converging to $f : M \to R$ such that f_n is cliquish but f is not cliquish.*

Proof. Let $A = \{a_1, a_2, \ldots\}$, $B = \{b_1, b_2, \ldots\}$ be countable subsets of R such that $A \cap B = \emptyset$, $\bar{A} = \bar{B} = R$. Denote $M = A \cup B$, $A_n = \{a_1, \ldots, a_n\}$, $B_n = \{b_1, \ldots, b_n\}$ for each $n \in N$. Define the sequence $\{f_n\}_{n=1}^{\infty}$ of functions $f_n : M \to R$ by

$$f_n = \begin{cases} \chi_{A_n} & \text{if } n \text{ is even}, \\ \chi_{(M - B_n)} & \text{if } n \text{ is odd}. \end{cases}$$

Obviously $f_n \to f = \chi_A$. Let $\varepsilon > 0$, $m \in \{0, 1, 2, \ldots\}$. Denote $p = m + 2$. Then for $x \in A$ we have $|f_{m+p-1}(x) - f(x)| = 0$, and for $x \in B$ we have $|f_{m+p}(x) - f(x)| = 0$, i.e.

$$\forall x \in M : \min\{|f_{m+1}(x) - f(x)|, \ldots, |f_{m+p}(x) - f(x)|\} < \varepsilon .$$

Hence $\{f_n\}_{n=1}^{\infty}$ quasi-uniformly converges to f. We now show that f_n $(n = 1, 2, \ldots)$ are cliquish functions. Let $n \in N$. Let $x_0 \in M$. Denote $\gamma = \min\{|x_0 - x| : x \in A_n \cup B_n,\ x \neq x_0\}$. Let U be an open set such that $x_0 \in U$. Let $\varepsilon > 0$. Denote $U_0 = (x_0, x_0 + \gamma) \cap U$. Then U_0 is open, $\emptyset \neq U_0 \subset U$ and

$$\forall x, x' \in U_0 : |f_n(x) - f_n(x')| = 0 < \varepsilon .$$

Since for each open set V we have

$$V \neq \emptyset \Rightarrow V \cap A \neq \emptyset \neq V \cap B ,$$

f is not cliquish.

Definition 1. *A family $\mathscr{A}$ of sets has the finite intersection property if the intersection of every finite subfamily of $\mathscr{A}$ is nonempty. A centred family is a family of sets having the finite intersection property.*

Definition 2. *An open almost-base for a space X is a family $\mathscr{A}$ of open subsets of X such that every nonempty open subset of X contains some nonempty $A \in \mathscr{A}$.*

Definition 3. *Let $\{\mathscr{A}_n\}_{n=1}^{\infty}$ be a sequence of open families in a space X (an open family is a family consisting of open sets). The sequence $\{\mathscr{A}_n\}_{n=1}^{\infty}$ is said to be countably complete if for every centred sequence of sets $\{A_{n_k}\}_{k=1}^{\infty}$, where $A_{n_k} \in \mathscr{A}_{n_k}$, the set $\bigcap_{k \in N} \bar{A}_{n_k}$ is nonempty.*

Definition 4. *A space X is said to be almost countably complete if there exists a countably complete sequence of open almost-bases for X (see [2]).*

Remark. Every locally compact Hausdorff space is a regular almost countably complete space. Every complete metric space is a regular almost countably complete space (see [2]).

Theorem. *Let X be a regular almost countably complete space and let (Y, d) be a metric space. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of cliquish functions $f_n : X \to Y$ quasi-uniformly converging to $f : X \to Y$. Then f is cliquish.*

Proof. We show that f is cliquish at any point $x_0 \in X$. Let U be an open set such that $x_0 \in U$. Let $\varepsilon > 0$. We now show that there exists a sequence of nonempty open sets $U_n \subset U$ such that

(i) $$\bigcap_{n \in N} U_n \neq \emptyset ,$$

(ii) $$\forall n \in N : \operatorname{diam} f_n(U_n) < \varepsilon/6 .$$

Let $\{\mathscr{B}_n\}_{n=1}^{\infty}$ be a countably complete sequence of open almost-bases for X. Since f_1 is cliquish at x_0, there exists a nonempty set $U_1 \in \mathscr{B}_1$ such that $U_1 \subset U$, $\operatorname{diam} f_1(U_1) < \varepsilon/6$. Suppose $U_1, \ldots, U_n$ have been constructed. Let $y \in U_n$. Since X is regular, there exists a closed neighbourhood W at y, such that $W \subset U_n$. Since f_{n+1} is cliquish at y, there exists a nonempty open set $U_{n+1} \in \mathscr{B}_{n+1}$ such that $U_{n+1} \subset \operatorname{int} W$, $\operatorname{diam} f_{n+1}(U_{n+1}) < \varepsilon/6$. Then

$$0 \neq U_{n+1} \subset \bar{U}_{n+1} \subset W \subset U_n \subset U , \quad U_n \in \mathscr{B}_n .$$

Since $\{U_{n+1}\}_{n=1}^{\infty}$ is centred,

$$\emptyset \neq \bigcap_{n \in N} \bar{U}_{n+1} \subset \bigcap_{n \in N} U_n .$$

Let $y \in \bigcap_{n \in N} U_n$. Since $f_n \to f$, we have

(1) $$\exists m \in N \ \forall n \geqq m : d(f(y), f_n(y)) < \varepsilon/6 .$$

Since $\{f_n\}_{n=1}^{\infty}$ quasi-uniformly converges to f, we have $\exists p \in N \ \forall x \in X$:

$$\min \{d(f_{m+1}(x), f(x)), \ldots, d(f_{m+p}(x), f(x))\} < \varepsilon/6 .$$

Denote $U_0 = \bigcap_{n=1}^{m+p} U_n$. Let $x \in U_0$. We now show that $d(f(x), f(y)) < \varepsilon/2$. Then obviously $\forall x, x' \in U_0 : d(f(x), f(x')) < \varepsilon$.

Let $j \in \{1, \ldots, p\}$ be such that $d(f_{m+j}(x), f(x)) < \varepsilon/6$. Then by (ii) we have $d(f_{m+j}(x), f_{m+j}(y)) < \varepsilon/6$, and by (1) we obtain $d(f_{m+j}(y), f(y)) < \varepsilon/6$, therefore $d(f(x), f(y)) \leqq d(f_{m+j}(x), f(x)) + d(f_{m+j}(x), f_{m+j}(y)) + d(f_{m+j}(y), f(y)) < \varepsilon/2$.

Remark. For a different proof of this theorem, see [1].

Corollary. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of cliquish functions $f_n : R \to R$ quasi-uniformly converging to $f : R \to R$. Then f is a cliquish function.

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