

ON A PRODUCT OF METRIC SPACES

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Introduction

There is a natural way of introducing an algebraic structure on a product of algebraic structures of the same type. For example, if $(A, +)$ and (B, \cdot) are groups, then $(A \times B, *)$, where $(a_1, b_1) * (a_2, b_2) = (a_1 + a_2, b_1 \cdot b_2)$ is a group as well. The application of this method to a collection $\{(A_i, d_i)\}_{i \in T}$ of metric spaces yields a mapping $(\varrho_d(x, y))(t) = d_i(x(t), y(t))$ which need not be a metric on $\prod_{i \in T} A_i$, since its values are in R^T . However, a metric can be obtained from that mapping by composing it with a suitable $f: R^T \rightarrow R$. In fact, the usual metrics on product spaces (as $\sqrt{\varrho^2 + \sigma^2}$, $\max(\varrho, \sigma)$, $\varrho + \sigma$, the Fréchet metric) can all be described in this way. Therefore it seems useful to investigate the set $\mathcal{M}(T)$ of all such mappings $f: R^T \rightarrow R$. A subset of $\mathcal{M}(T)$ (consisting of all nonnegative, monotone, subadditive mappings vanishing exactly at the constant zero function) is studied in [2]. In [1], the set $\mathcal{M}(T)$ is described in the special case when T has only one element. In the present paper we give a complete characterization of $\mathcal{M}(T)$ in the general case when T is an arbitrary set, and establish a necessary and sufficient condition for $f \circ \varrho_d$ to metrize the product topology. Theorem 2.11 was inspired by a suggestion of T. Šalát.

1. Preliminary considerations

1.1. Definition. Let T be a set. Let $d = (d_i)_{i \in T}$ be a collection of mappings $d_i: A_i^2 \rightarrow B_i$, where $(A_i)_{i \in T}$, $(B_i)_{i \in T}$ are collections of sets. Define a mapping $\varrho_d: \left(\prod_{i \in T} A_i\right)^2 \rightarrow \prod_{i \in T} B_i$ by $(\varrho_d(x, y))(t) = d_i(x(t), y(t))$ for each $x, y \in \prod_{i \in T} A_i$, $t \in T$. Define a mapping $\sigma_d: \left(\prod_{i \in T} A_i\right)^3 \rightarrow \left(\prod_{i \in T} B_i\right)^3$ by $\sigma_d(x, y, z) = (\varrho_d(x, y), \varrho_d(x, z), \varrho_d(y, z))$.

$\varrho_a(y, z)$ for each $x, y, z \in \prod_{i \in T} A_i$. Denote $E_a = \left\{ \varrho_a(x, x) : x \in \prod_{i \in T} A_i \right\}$, and $F_a = \left\{ \varrho_a(x, y) : x, y \in \prod_{i \in T} A_i, x \neq y \right\}$.

1.2. Theorem. Let $B \supset \text{Im } \varrho_a$ be a set (where $\text{Im } f = \{f(x) : x \in X\}$ for each mapping $f: X \rightarrow Y$). Let $f: B \rightarrow R$. Then $f \circ \varrho_a$ is a metric if and only if the following three conditions are satisfied:

- (1)
$$E_a \cap F_a = \emptyset,$$
- (2)
$$\forall x \in \text{Im } \varrho_a : f(x) = 0 \Leftrightarrow x \in E_a,$$
- (3)
$$\forall x, y, z \in \text{Im } \varrho_a : (x, y, z) \in \text{Im } \sigma_a \Rightarrow f(x) \leq f(y) + f(z).$$

Proof. Necessity. Suppose that $a \in E_a \cap F_a$. Then $\exists x, y, z \in \prod_{i \in T} A_i, y \neq z$: $\varrho_a(x, x) = a = \varrho_a(y, z)$, therefore $0 = (f \circ \varrho_a)(x, x) = f(\varrho_a(x, x)) = f(\varrho_a(y, z)) = (f \circ \varrho_a)(y, z)$, a contradiction. This shows that $E_a \cap F_a = \emptyset$. Let $x \in \text{Im } \varrho_a$. Then $\exists a, b \in \prod_{i \in T} A_i : x = \varrho_a(a, b)$, therefore $0 = f(x) = f(\varrho_a(a, b)) = (f \circ \varrho_a)(a, b) \Leftrightarrow a = b \Leftrightarrow x = \varrho_a(a, b) \in E_a$.

Let $(x, y, z) \in \text{Im } \sigma_a$. Then $\exists a, b, c \in \prod_{i \in T} A_i : x = \varrho_a(a, b), y = \varrho_a(a, c), z = \varrho_a(b, c)$, hence $f(x) = f(\varrho_a(a, b)) = (f \circ \varrho_a)(a, b) \leq (f \circ \varrho_a)(a, c) + (f \circ \varrho_a)(b, c) = f(y) + f(z)$.

Sufficiency. Let $x, y \in \prod_{i \in T} A_i$. Then $0 = (f \circ \varrho_a)(x, y) = f(\varrho_a(x, y)) \Leftrightarrow \varrho_a(x, y) \in E_a \Leftrightarrow x = y$. Let $x, y, z \in \prod_{i \in T} A_i$. Then $\sigma_a(x, y, z) \in \text{Im } \sigma_a$, hence $(f \circ \varrho_a)(x, y) = f(\varrho_a(x, y)) \leq f(\varrho_a(x, z)) + f(\varrho_a(y, z)) = (f \circ \varrho_a)(x, z) + (f \circ \varrho_a)(y, z)$.

1.3. Corollary. Let $h = (h_i)_{i \in T}$ be a collection of mappings $h_i: C_i^2 \rightarrow D_i$, where $(C_i)_{i \in T}, (D_i)_{i \in T}$ are collections of sets. Let $E_h = E_a, E_h \cap F_h = \emptyset, \text{Im } \varrho_h \subset \text{Im } \varrho_a, \text{Im } \sigma_h \subset \text{Im } \sigma_a$. Let $B \supset \text{Im } \varrho_a$ be a set. Let $f: B \rightarrow R$ be a mapping such that $f \circ \varrho_a$ is a metric. Then $f \circ \varrho_h$ is a metric.

1.4. Proposition. Let $f: A \rightarrow R$ and $g: B \rightarrow R$ be mappings, where $A, B \supset \text{Im } \varrho_a$. Define a mapping $f+g: (A \cap B) \rightarrow R$ by $(f+g)(x) = f(x) + g(x)$ for each $x \in A \cap B$. Define a mapping $\max(f, g): (A \cap B) \rightarrow R$ by $\max(f, g)(x) = \max(f(x), g(x))$ for each $x \in A \cap B$. Let $f \circ \varrho_a$ and $g \circ \varrho_a$ be metrics. Then $(f+g) \circ \varrho_a, \max(f, g) \circ \varrho_a$ are metrics.

Proof. Let $x \in \text{Im } \varrho_a$. Then $0 = (f+g)(x) = f(x) + g(x) \Leftrightarrow f(x) = 0 \& g(x) = 0 \Leftrightarrow x \in E_a$. Let $(x, y, z) \in \text{Im } \sigma_a$. Then $(f+g)(x) = f(x) + g(x) \leq f(y) + f(z) + g(y) + g(z) = (f+g)(y) + (f+g)(z)$. Then by 1.2, $(f+g) \circ \varrho_a$ is a metric.

Let $x \in \text{Im } \varrho_d$. Then $0 = (\max(f, g))(x) = \max(f(x), g(x)) \Leftrightarrow f(x) = 0 \& g(x) = 0 \Leftrightarrow x \in E_d$. Let $(x, y, z) \in \text{Im } \sigma_d$. Then $f(x) \leq f(y) + f(z) \leq \max(f(y), g(y)) + \max(f(z), g(z))$, $g(x) \leq g(y) + g(z) \leq \max(f(y), g(y)) + \max(f(z), g(z))$, i.e. $(\max(f, g))(x) = \max(f(x), g(x)) \leq \max(f(y), g(y)) + \max(f(z), g(z)) = (\max(f, g))(y) + (\max(f, g))(z)$. Then by 1.2, $\max(f, g)$ is a metric.

1.5. Proposition. Let $\{f_i\}_{i=1}^{\infty}$ be a sequence of mappings $f_i: C_i \rightarrow R$, where $C_i \supset \text{Im } \varrho_d$. Let $\{f_i(x)\}_{i=1}^{\infty}$ converge for each $x \in \bigcap_{i=1}^{\infty} C_i$. Define a mapping $\lim_{i \rightarrow \infty} f_i: \bigcap_{i=1}^{\infty} C_i \rightarrow R$ by $(\lim_{i \rightarrow \infty} f_i)(x) = \lim_{i \rightarrow \infty} f_i(x)$ for each $x \in \bigcap_{i=1}^{\infty} C_i$. Let $\forall x \in F_d: (\lim_{i \rightarrow \infty} f_i)(x) \neq 0$. Let $f_i \circ \varrho_d$ be a metric for every $i \in N$. Then $(\lim_{i \rightarrow \infty} f_i) \circ \varrho_d$ is a metric.

Proof. Let $(x, y, z) \in \text{Im } \sigma_d$. Then $(\lim_{i \rightarrow \infty} f_i)(x) = \lim_{i \rightarrow \infty} f_i(x) \leq \lim_{i \rightarrow \infty} (f_i(y) + f_i(z)) = \lim_{i \rightarrow \infty} f_i(y) + \lim_{i \rightarrow \infty} f_i(z) = (\lim_{i \rightarrow \infty} f_i)(y) + (\lim_{i \rightarrow \infty} f_i)(z)$. Then by 1.2, $(\lim_{i \rightarrow \infty} f_i) \circ \varrho_d$ is a metric.

1.6. Corollary. Let $\sum_{i=1}^{\infty} f_i$ be a series of functions $f_i: C_i \rightarrow R$, where $C_i \supset \text{Im } \varrho_d$. Let $\sum_{i=1}^{\infty} f_i(x)$ be convergent for each $x \in \bigcap_{i=1}^{\infty} C_i$. Let $f_i \circ \varrho_d$ be a metric for all $i \in N$. Then $(\lim_{n \rightarrow \infty} \sum_{i=1}^n f_i) \circ \varrho_d = (\sum_{i=1}^{\infty} f_i) \circ \varrho_d$ is a metric.

Proof. By 1.4 $(\sum_{i=1}^n f_i) \circ \varrho_d$ is a metric for any $n \in N$. Let $x \in F_d$. Then $\forall i \in N: f_i(x) > 0$, therefore $\forall n \in N: \sum_{i=1}^n f_i(x) \geq f_1(x)$, i.e. $(\lim_{n \rightarrow \infty} \sum_{i=1}^n f_i)(x) = \lim_{n \rightarrow \infty} \sum_{i=1}^n f_i(x) \geq f_1(x) > 0$. Then by 1.5 $(\lim_{n \rightarrow \infty} \sum_{i=1}^n f_i) \circ \varrho_d$ is a metric.

1.7. Proposition. Let $f = (f_t)_{t \in I}$ be a collection of functions $f_t: C_t \rightarrow R$, where $C_t \supset \text{Im } \varrho_d$ and $I \neq \emptyset$. Let the set $A_x = \{f_t(x): t \in I\}$ be bounded above for each $x \in \bigcap_{t \in I} C_t$. Define a function $\sup f: \bigcap_{t \in I} C_t \rightarrow R$ by $(\sup f)(x) = \sup A_x$ for each $x \in \bigcap_{t \in I} C_t$. Let $f_t \circ \varrho_d$ be a metric for every $t \in I$. Then $(\sup f) \circ \varrho_d$ is a metric.

Proof. Let $x \in F_d$. Then $\forall t \in I: f_t(x) > 0$, thus $A_x \subset (0, \infty)$, i.e. $\sup A_x > 0$. Hence $\forall x \in \text{Im } \varrho_d: (\sup f)(x) = 0 \Leftrightarrow x \in E_d$. Let $(x, y, z) \in \text{Im } \sigma_d$. Then $\forall t \in I: f_t(x) \leq f_t(y) + f_t(z) \leq \sup A_y + \sup A_z$. Then by 1.2 it follows that $(\sup f) \circ \varrho_d$ is a metric.

2. Characterization of $\mathcal{M}(T)$

2.1. Definition. Let T be a nonempty set. Suppose R^T is ordered coordinate-wise, i.e. $x \leq y$ ($x < y$) if and only if $x(t) \leq y(t)$ ($x(t) < y(t)$) for each $x, y \in R^T$, $t \in T$. Define a function $\Theta: T \rightarrow R$ by $\Theta(t) = 0$ for each $t \in T$. Denote $T^+ = \{x \in R^T: x \geq \Theta\}$. Denote by $\mathcal{M}(T)$ the set of all functions $f: T^+ \rightarrow R$ such that $f \circ \rho_a$ is a metric for every collection of metrics $d = (d_t)_{t \in T}$.

2.2. Proposition. Let $f: T^+ \rightarrow R$ be a function such that

- (i) $f(\Theta) = 0$,
- (ii) $\exists a > 0 \forall x \in T^+, x \neq \Theta: f(x) \in \langle a, 2a \rangle$.

Then $f \in \mathcal{M}(T)$.

Proof. Let $d = (d_t)_{t \in T}$ be a collection of metrics $d_t: M_t \times M_t \rightarrow R$. Then $E_d = \{\Theta\}$, $\Theta \notin F_d$, hence $E_d \cap F_d = \emptyset$. Let $x \in \text{Im } \rho_a$. Then $f(x) = 0 \Leftrightarrow x = \Theta \Leftrightarrow x \in E_d$. Let $x, y, z \in \text{Im } \rho_a$, $(x, y, z) \in \text{Im } \sigma_a$. If $x = \Theta$, then $f(x) = 0 \leq f(y) + f(z)$. If $y = \Theta$, then $x = z$, hence $f(x) = 0 + f(z) = f(y) + f(z)$. If $z = \Theta$, then $x = y$, hence $f(x) = f(y) + 0 = f(y) + f(z)$. If $\Theta \notin \{x, y, z\}$, then $f(x) \leq 2a = a + a \leq f(y) + f(z)$. Then by 1.2 we see that $f \circ \rho_a$ is a metric.

2.3. Lemma. Let $f \in \mathcal{M}(T)$. Then

$$\forall x \in T^+: f(x) = 0 \Leftrightarrow x = \Theta.$$

Proof. Let $\rho: S \times S \rightarrow R$ be a metric such that $\forall a \geq 0 \exists x, y \in S: \rho(x, y) = a$ (for example $S = R$, $\rho(u, v) = |u - v|$ for every $u, v \in R$, $x = a$, $y = 0$). Define a collection of metrics $d = (d_t)_{t \in T}$ by $d_t = \rho$ for each $t \in T$. Then $\text{Im } \rho_a = T^+$, $E_d = \{\Theta\}$. Hence by 1.2 it follows that $\forall x \in T^+: f(x) = 0 \Leftrightarrow x \in E_d \Leftrightarrow x = \Theta$.

2.4. Lemma. Let $f \in \mathcal{M}(T)$. Then $\forall x, y, z \in T^+$:

$$(x \leq y + z \ \& \ y \leq x + z \ \& \ z \leq x + y) \Rightarrow f(x) \leq f(y) + f(z).$$

Proof. Let $\rho: S \times S \rightarrow R$ be a metric such that $\forall a, b, c \geq 0, a \leq b + c, b \leq a + c, c \leq a + b \exists x, y, z \in S: \rho(x, y) = a, \rho(y, z) = c, \rho(x, z) = b$ (for example $S = R \times R$, $\rho(u, v) = \|u - v\|$ for each $u, v \in R \times R$, $x = (a/2, 0)$, $y = (-a/2, 0)$, $z = ((c^2 - b^2)/(2a), (\sqrt{((a+b+c) \cdot (a+b-c) \cdot (a-b+c) \cdot (-a+b+c))})/(2a))$ for $a \neq 0$, $z = (b, 0)$ for $a = 0$). Define a collection of metrics $d = (d_t)_{t \in T}$ by $d_t = \rho$ for all $t \in T$. Let $x, y, z \in T^+$, $x \leq y + z$, $y \leq x + z$, $z \leq x + y$. Since $\{(x, y, z) \in (T^+)^3: x \leq y + z, y \leq x + z, z \leq x + y\} \subset \text{Im } \sigma_a$, by 1.2 we obtain $f(x) \leq f(y) + f(z)$.

2.5. Lemma. Let $f \in \mathcal{M}(T)$. Then

- (i) $\forall x, y \in T^+: f(x + y) \leq f(x) + f(y)$,
- (ii) $\forall x, y \in T^+: x \leq 2y \Rightarrow f(x) \leq 2f(y)$.

Proof. Let $x, y \in T^+$. Since $(x + y) \leq x + y$, $x \leq (x + y) + y$, $y \leq (x + y) + x$, by

2.4 we have $f(x+y) \cong f(x)+f(y)$. Let $x, y \in T^+$, $x \cong 2y$. Since $x \cong y+y$, $y \cong x+y$, by 2.4 we get $f(x) \cong f(y)+f(y) = 2f(y)$.

2.6. Theorem. Let $f: T^+ \rightarrow R$. Then $f \in \mathcal{M}(T)$ if and only if

(i) $\forall x \in T^+ : f(x)=0 \Leftrightarrow x = \Theta$,

(ii) $\forall x, y, z \in T^+ : (x \cong y+z \ \& \ y \cong x+z \ \& \ z \cong x+y) \Rightarrow f(x) \cong f(y)+f(z)$.

Proof. Sufficiency. Let $d = (d_i)_{i \in T}$ be a collection of metrics $d_i: M_i \times M_i \rightarrow R$. Then $E_d = \{\Theta\}$, $\Theta \notin F_d$, therefore $E_d \cap F_d = \emptyset$. Let $x \in \text{Im } \rho_d \subset T^+$. Then $f(x)=0 \Leftrightarrow x = \Theta \Leftrightarrow x \in E_d$. Let $(x, y, z) \in \text{Im } \sigma_d$. Then $x \cong y+z$, $y \cong x+z$, $z \cong x+y$, hence $f(x) \cong f(y)+f(z)$.

Necessity. By 2.3 and 2.4.

2.7. Proposition. Let $f, g \in \mathcal{M}(T)$. Then $f+g \in \mathcal{M}(T)$, $\max(f, g) \in \mathcal{M}(T)$. (See 1.4)

2.8. Definition. Let (M, d) be a metric space and let Ω denote the first uncountable ordinal number. The transfinite sequence

$$(1) \quad \{a_\xi\}_{\xi < \Omega}$$

of elements of the space M is said to be convergent and to have a limit $a \in M$ if for each $\varepsilon > 0$ there exists an ordinal number $\alpha < \Omega$ such that $d(a_\xi, a) < \varepsilon$ whenever $\alpha \leq \xi < \Omega$. If (1) has a limit a , we write $\lim_{\xi \rightarrow \Omega} a_\xi = a$ (or briefly $a_\xi \rightarrow a$). (See [3], [4].)

2.9. Definition. Let X be a set and let (Y, d) be a metric space. The transfinite sequence

$$(2) \quad \{f_\xi\}_{\xi < \Omega}$$

of functions $f_\xi: X \rightarrow Y$ is said to be convergent and to have a limit function $f: X \rightarrow Y$ if for each $x \in X$ we have $\lim_{\xi \rightarrow \Omega} f_\xi(x) = f(x)$. If (2) has a limit function f , we write

$\lim_{\xi \rightarrow \Omega} f_\xi = f$ (or briefly $f_\xi \rightarrow f$). (See [3], [4].)

2.10. Lemma. Let (M, d) be a metric space, $a_\xi \in M$ ($\xi < \Omega$) and $a_\xi \rightarrow a$. Then there exists an ordinal number $\alpha < \Omega$ such that $a_\xi = a$ for each ξ with $\alpha \leq \xi < \Omega$. (See [4; lemma 1].)

2.11. Theorem. Let $f_\xi \in \mathcal{M}(T)$ ($\xi < \Omega$) and let $f_\xi \rightarrow f$. Then $f \in \mathcal{M}(T)$.

Proof. Let $a \in T^+$. Since $f_\xi \rightarrow f$, by 2.10 there exists an ordinal number $\alpha = \alpha(a) < \Omega$ such that $f_\xi(a) = f(a)$ whenever $\alpha \leq \xi < \Omega$. Then $0 = f(a) = f_a(a) \Leftrightarrow a = \Theta$.

Let $a, b, c \in T^+$, $a \cong b+c$, $b \cong a+c$, $c \cong a+b$. Since $f_\xi \rightarrow f$, by 2.10 there exists an ordinal number $\beta = \beta(a, b, c) < \Omega$ such that $f_\xi(a) = f(a)$, $f_\xi(b) = f(b)$, $f_\xi(c) = f(c)$ for each ξ with $\beta \leq \xi < \Omega$. Then $f(a) = f_\beta(a) \cong f_\beta(b) + f_\beta(c) = f(b) + f(c)$.

2.12. Proposition. Let $\{f_i\}_{i=1}^{\infty}$ be a sequence of functions $f_i \in \mathcal{M}(T)$ such that the sequence $\{f_i(x)\}_{i=1}^{\infty}$ converges for each $x \in T^+$. Let $\forall x \in T^+$, $x \neq \Theta$: $(\lim_{i \rightarrow \infty} f_i)(x) \neq 0$. Then $\lim_{i \rightarrow \infty} f_i \in \mathcal{M}(T)$. (See 1.5)

2.13. Proposition. Let $\sum_{i=1}^{\infty} f_i$ be a series of functions $f_i \in \mathcal{M}(T)$ such that the series $\sum_{i=1}^{\infty} f_i(x)$ converges for each $x \in T^+$. Then $(\lim_{n \rightarrow \infty} \sum_{i=1}^n f_i) \in \mathcal{M}(T)$. (See 1.6)

2.14. Proposition. Let $f = (f_i)_{i \in I}$ be a collection of functions $f_i \in \mathcal{M}(T)$ such that the set $\{f_i(x) : i \in I\}$ is bounded above. Then $\sup f \in \mathcal{M}(T)$. (See 1.7)

2.15. Theorem. Let $f \in \mathcal{M}(T)$. Then f is continuous if and only if f is continuous at the point Θ .

Proof. Denote by \mathcal{T} the usual topology on R . Denote by \mathcal{S}_T the product topology on R^T . Let $\varepsilon > 0$. Then

$$\exists U \in \mathcal{S}_T, \Theta \in U \forall x \in U \cap T^+ : f(x) < \varepsilon.$$

Therefore there exists a base element $V \subset U$, $\Theta \in V$, i.e. $\exists F \subset T$, F is finite nonempty $\forall t \in F \exists U_t \in \mathcal{T}, 0 \in U_t : V = \bigcap_{t \in F} \pi_t^{-1}(U_t)$, where π_t is the projection from R^T into R , i.e. $\pi_t(x) = x(t)$ for each $x \in R^T$. Let $t \in F$. Then $\exists \gamma_t > 0 : (-\gamma_t, \gamma_t) \subset U_t$. Denote $\gamma = \min_{t \in F} \gamma_t$. Then $\bigcap_{t \in F} \pi_t^{-1}((-\gamma, \gamma)) \subset V$, therefore $\forall x \in T^+ : (\forall t \in F : x(t) < \gamma) \Rightarrow f(x) < \varepsilon$.

Let $x \in T^+$, $x \neq \Theta$. Denote $\delta = \gamma/2$. Let $y \in T^+$ be a function such that $\forall t \in F : |x(t) - y(t)| < \delta$.

Define a function $z : T \rightarrow R$ by

$$\begin{aligned} z(t) &= \min(\delta, x(t) + y(t)) \text{ for } t \in F, \\ z(t) &= x(t) + y(t) \text{ for } t \in T - F. \end{aligned}$$

Then $z \in T^+$, $x \leq y + z$, $y \leq x + z$, $z \leq x + y$, $\forall t \in F : z(t) < \gamma$, hence $|f(x) - f(y)| \leq f(z) < \varepsilon$. Therefore $\forall x \in T^+$, $x \neq \Theta \forall \varepsilon > 0 \exists W \in \mathcal{S}_T, x \in W \forall y \in W \cap T^+ : |f(x) - f(y)| < \varepsilon$ ($W = \bigcap_{t \in F} \pi_t^{-1}(S(x(t), \delta))$) and since, by the hypothesis, f is continuous at the point Θ , f is continuous.

2.16. Lemma. Let $f \in \mathcal{M}(T)$ be continuous. Then

$$\forall \varepsilon > 0 \exists x \in T^+, x > \Theta : f(x) < \varepsilon.$$

Proof. Let $\varepsilon > 0$. Since f is continuous at the point Θ , we have $\exists U \in \mathcal{S}_T$, $\Theta \in U$ $\forall x \in U \cap T^+$: $f(x) < \varepsilon$. Since $U \in \mathcal{S}_T$ and $\Theta \in U$, $\exists \delta > 0$ $\exists F \subset T$, F is finite nonempty: $\bigcap_{t \in F} \pi_t^{-1}(S(0, \delta)) \subset U$. Define a function $x: T \rightarrow R$ by $x(t) = \delta/2$ for each $t \in T$. Then $x \in U \cap T^+$, therefore $f(x) < \varepsilon$.

2.17. Proposition. Let T be a finite set. Let $f \in \mathcal{M}(T)$. Then f is continuous if and only if

$$\forall \varepsilon > 0 \exists x \in T^+, \quad x > \Theta: f(x) < \varepsilon.$$

Proof. Sufficiency. Let $\varepsilon > 0$. Then for $\varepsilon > 0$ there is $a \in T^+$, $a > \Theta: f(a) < \varepsilon/2$. Since $\forall x \in T^+: x \leq 2a \Rightarrow f(x) \leq 2f(a) < \varepsilon$, hence for $U = \bigcap_{t \in T} \pi_t^{-1}(S(0, \min_{t \in T} a(t)))$ there holds $U \in \mathcal{S}_T$, $\Theta \in U$, $\forall x \in U \cap T^+: f(x) < \varepsilon$, therefore f is continuous at the point Θ and by 2.15 f is continuous. Necessity follows from 2.16.

2.18. Example. Let $f: \{0, 1\}^+ \rightarrow R$ be defined as follows:

$$\begin{aligned} f(\{(0, x), (1, y)\}) &= 1 \quad \text{for } x \neq 0, \\ f(\{(0, x), (1, y)\}) &= \min(1, y) \quad \text{for } x = 0. \end{aligned}$$

Then $f \in \mathcal{M}(\{0, 1\})$, f is not continuous and we have

$$\forall \varepsilon > 0 \exists x \in \{0, 1\}^+, \quad x \neq \Theta: f(x) < \varepsilon$$

(for example $x = \{(0, 0), (1, \min(1/2, \varepsilon/2))\}$).

2.19. Corollary. Let T be a finite set. Let $f \in \mathcal{M}(T)$. Then f is not continuous if and only if

$$\exists \eta > 0 \forall x \in T^+, \quad x > \Theta: f(x) \geq \eta.$$

2.20. Lemma. Let $f: S \rightarrow T$ be a bijective mapping. Define a mapping $f^*: T^+ \rightarrow S^+$ by $f^*(a) = a \circ f$ for all $a \in T^+$. Let $g: S^+ \rightarrow R$. Then $g \in \mathcal{M}(S)$ if and only if $(g \circ f^*) \in \mathcal{M}(T)$.

Proof. Necessity. Let $a \in T^+$. Then $0 = (g \circ f^*)(a) = g(f^*(a)) \Leftrightarrow f^*(a) = \Theta \Leftrightarrow \forall t \in S: a(f(t)) = 0 \Leftrightarrow \forall t \in T: a(t) = 0 \Leftrightarrow a = \Theta$.

Let $a, b, c \in T^+$, $a \leq b + c$, $b \leq a + c$, $c \leq a + b$. Then $f^*(a) \leq f^*(b) + f^*(c)$, $f^*(b) \leq f^*(a) + f^*(c)$, $f^*(c) \leq f^*(a) + f^*(b)$, hence $(g \circ f^*)(a) = g(f^*(a)) \leq g(f^*(b)) + g(f^*(c)) = (g \circ f^*)(b) + (g \circ f^*)(c)$. Then by 2.6. we obtain $(g \circ f^*) \in \mathcal{M}(T)$.

Sufficiency. Since $f^{-1}: T \rightarrow S$ is a bijective mapping, we have $g = (g \circ f^*) \circ (f^{-1})^* \in \mathcal{M}(S)$.

2.21. Lemma. Let $S \subset T$ be a nonempty set, $i: S \rightarrow T$, $i(x) = x$. Define a mapping $i_*: S^+ \rightarrow T^+$ for each $a \in S^+$ by $(i_*(a))(t) = a(t)$ for $t \in S$ and $(i_*(a))(t) = 0$ for $t \in T - S$. Let $f \in \mathcal{M}(T)$. Then $(f \circ i_*) \in \mathcal{M}(S)$.

Proof. Let $a \in S^+$. Then $0 = (f \circ i_*)(a) = f(i_*(a)) \Leftrightarrow i_*(a) = \Theta \Leftrightarrow a = \Theta$.

Let $a, b, c \in S^+$, $a \leq b + c$, $b \leq a + c$, $c \leq a + b$. Then $i_*(a) \leq i_*(b) + i_*(c)$, $i_*(b) \leq i_*(a) + i_*(c)$, $i_*(c) \leq i_*(b) + i_*(a)$, therefore $(f \circ i_*)(a) = f(i_*(a)) \leq f(i_*(b)) + f(i_*(c)) = (f \circ i_*)(b) + (f \circ i_*)(c)$. Then $(f \circ i_*) \in \mathcal{M}(S)$ (by 2.6).

2.22. Proposition. Let S be a nonempty set. Let $f: S \rightarrow T$ be an injective mapping. Define a mapping $i: \text{Im}f \rightarrow T$ by $i(x) = x$. Let $g: \text{Im}f \rightarrow S$ be a bijective mapping such that $f = i \circ g^{-1}$. Define a mapping $f^*: S^+ \rightarrow T^+$ by $f^* = i^* \circ g^*$. Let $h \in \mathcal{M}(T)$. Then $(h \circ f^*) \in \mathcal{M}(S)$.

Proof. Since by 2.21 $(h \circ i^*) \in \mathcal{M}(\text{Im}f)$, it follows from 2.20 that $h \circ f^* = (h \circ i^*) \circ g^* \in \mathcal{M}(S)$.

2.23. Remark. Let \mathcal{K} and \mathcal{S} be categories whose objects are nonempty sets and morphisms are injective mappings and mappings, respectively. Assign the set $\mathcal{M}(T)$ to each object T of \mathcal{K} . For every morphism $f: S \rightarrow T$ of the category \mathcal{K} define a mapping $\mathcal{M}(f): \mathcal{M}(T) \rightarrow \mathcal{M}(S)$ by $(\mathcal{M}(f))(g) = g \circ f^*$ whenever $g \in \mathcal{M}(T)$. Thus we have described a contravariant functor $\mathcal{M}: \mathcal{K} \rightarrow \mathcal{S}$.

3. Metrization of the product topology

3.1. Lemma. Let $d = (d_i)_{i \in T}$ be a collection of metrics $d_i: M_i^2 \rightarrow R$. Let $f \in \mathcal{M}(T)$.

Denote by \mathcal{T}_S the product topology on $\prod_{i \in T} M_i$ and denote by \mathcal{T}_f the topology generated by the metric $f \circ \varrho_d$. Then $\mathcal{T}_S \subset \mathcal{T}_f$.

Proof. Let $t \in T$. Let U_t be an open set in M_t . Let $x \in \pi_t^{-1}(U_t)$, where π_t is the projection from $\prod_{i \in T} M_i$ into M_t , i.e. $\pi_t(x) = x(t)$ for each $x \in \prod_{i \in T} M_i$. Then $x(t) \in U_t$, therefore $\exists \varepsilon > 0: S(x(t), \varepsilon) \subset U_t$. Define a function $a: T \rightarrow R$ by $a(t) = 2\varepsilon$ and $a(i) = 0$ for each $i \in T - \{t\}$.

Put $\delta = f(a)/2$. Let $y \in S(x, \delta) \in \mathcal{T}_f$. Then $(f \circ \varrho_d)(x, y) < \delta$, therefore $f(\varrho_d(x, y)) < \delta = f(a)/2$. By 2.6 we have $\forall b \in T^+: a \leq 2b \Rightarrow f(a) \leq 2f(b)$, or equivalently $\forall b \in T^+: f(b) < f(a)/2 \Rightarrow \neg(b \geq a/2)$.

Hence $\neg(\varrho_d(x, y) \geq a/2)$ and therefore, by definition a , we have $d_i(x(t), y(t)) = (\varrho_d(x, y))(t) < a(t)/2 = \varepsilon$. Therefore $y \in \pi_t^{-1}(S(x(t), \varepsilon)) \subset \pi_t^{-1}(U_t)$. Then

$$\forall x \in \pi_t^{-1}(U_t) \exists V \in \mathcal{T}_f, x \in V: V \subset \pi_t^{-1}(U_t) \quad (V = S(x, \delta)),$$

which implies $\mathcal{T}_S \subset \mathcal{T}_f$.

3.2. Proposition. Let $d = (d_i)_{i \in T}$ be a collection of metrics $d_i: M_i^2 \rightarrow R$. Put $H_d = \{t \in T: M'_t \neq \emptyset\}$ (where M' is the set of all accumulation points of the metric space (M, ϱ)). Let $F, H_d \subset F \subset T$, be such a set that $T - F$ is a finite set. Let $i: F \rightarrow T$ be a mapping defined by $i(x) = x$. Let $f \in \mathcal{M}(T)$. Let $f \circ i^*$ be a continuous mapping. Then $\mathcal{T}_S = \mathcal{T}_f$.

Proof. Since by 3.1 we have $\mathcal{T}_S \subset \mathcal{T}_f$, it is sufficient to prove that $\mathcal{T}_f \subset \mathcal{T}_S$.

Let $x \in \prod_{t \in T} M_t$ and $\varepsilon > 0$. The function $f \circ i_*$ is continuous at the point Θ , i.e. $\exists K \subset F$, $K \neq \emptyset$ finite $\exists \gamma > 0 \forall y \in F^+$: $(\forall t \in K: y(t) < \gamma) \Rightarrow (f \circ i_*)(y) < \varepsilon$.

The set $T - F$ is finite, this implies that there exists $\beta > 0$ such that $\forall t \in T - F \forall y \in M_t$, $y \neq x(t)$: $d_t(x(t), y) \geq \beta$. Denote $\delta = \min(\beta, \gamma)$ and $L = K \cup (T - F)$.

Put $V = \bigcap_{t \in L} \pi_t^{-1}(S(x(t), \delta))$. Then $V \in \mathcal{T}_S$ and $x \in V$. Let $y \in V$.

Then $\forall t \in T - F$: $(\varrho_d(x, y))(t) = 0$, this implies $i_*(\varrho_d(x, y)|_F) = \varrho_d(x, y)$. Since $\varrho_d(x, y)|_F \in F^+$ and $\forall t \in K$: $(\varrho_d(x, y)|_F)(t) = d_t(x(t), y(t)) < \gamma$, we have $(f \circ \varrho_d)(x, y) = f(\varrho_d(x, y)) = (f \circ i_*)(\varrho_d(x, y)|_F) < \varepsilon$, i.e. $y \in S(x, \varepsilon)$. Therefore

$\forall x \in \prod_{t \in T} M_t \forall \varepsilon > 0 \exists V \in \mathcal{T}_S: x \in V \subset S(x, \varepsilon)$, i.e. $\mathcal{T}_f = \mathcal{T}_S$.

3.3. Corollary. Let $d = (d_t)_{t \in T}$ be a collection of metrics $d_t: M_t^2 \rightarrow R$. Let $f \in \mathcal{M}(T)$ be a continuous mapping. Then $\mathcal{T}_S = \mathcal{T}_f$.

3.4. Proposition. Let $d = (d_t)_{t \in T}$ be a collection of metrics $d_t: M_t^2 \rightarrow R$. Denote $I_d = \{t \in T: \sup \text{Im } d_t \in R\}$. Let M_t be a nonempty set for each $t \in T$. Let $H_d \cap I_d$ be a finite set. Let $i: H_d \rightarrow T$ be a mapping defined by $i(x) = x$. Let $f \in \mathcal{M}(T)$ be a mapping such that $\mathcal{T}_S = \mathcal{T}_f$. Then $f \circ i_*$ is a continuous mapping.

Proof. If $H_d = \emptyset$, then the statement is true. Suppose $H_d \neq \emptyset$. Then by 2.21 $f \circ i_* \in \mathcal{M}(H_d)$, therefore it is sufficient to prove that $f \circ i_*$ is continuous at the point Θ . Since M_t is nonempty for all t in T , there exists x in $\prod_{t \in T} M_t$ such that $\forall t \in H_d: x(t) \in M_t$. Let $\varepsilon > 0$. Then $S(x, \varepsilon/2) \in \mathcal{T}_f \subset \mathcal{T}_S$, hence

$$\exists K \subset T, K \neq \emptyset \text{ finite } \exists \gamma > 0: \bigcap_{t \in K} \pi_t^{-1}(S(x(t), \gamma)) \subset S(x, \varepsilon/2).$$

Let F be a nonempty finite set such that $H_d \cap (K \cup I_d) \subset F \subset H_d$. Let $t \in F$. Since $x(t) \in M_t$, there exists $y_t \in M_t$ with $0 < d_t(x(t), y_t) < \gamma$. Put $\delta = \min_{t \in F} d_t(x(t), y_t)$. Let

$z \in H_d^+$, $z \in \bigcap_{t \in F} \pi_t^{-1}(S(0, \delta))$. Then $\forall t \in H_d - F \exists y_t \in M_t: z(t) \leq d_t(x(t), y_t)$. Define

a mapping $y: T \rightarrow \bigcup_{t \in T} M_t$ by $y(t) = y_t$ for $t \in H_d$, $y(t) = x(t)$ for $t \in T - H_d$. Then

$y \in \bigcap_{t \in K} \pi_t^{-1}(S(x(t), \gamma))$ and $i_*(z) \leq 2\varrho_d(x, y)$, hence

$$(f \circ i_*)(z) = f(i_*(z)) \leq 2f(\varrho_d(x, y)) = 2(f \circ \varrho_d)(x, y) < \varepsilon.$$

Therefore $\forall \varepsilon > 0 \exists F \subset H_d, F \neq \emptyset$ finite $\exists \delta > 0$

$$\forall z \in \bigcap_{t \in F} \pi_t^{-1}(S(0, \delta)): (f \circ i_*)(z) < \varepsilon,$$

i.e. $f \circ i_*$ is continuous at the point Θ .

3.5. Corollary. Let d_t be the usual metric on R for all $t \in T$. Let $f \in \mathcal{M}(T)$. Then $\mathcal{T}_S = \mathcal{T}_f$ if and only if f is continuous.

3.6. Theorem. Let T be a finite set. Let $d = (d_t)_{t \in T}$ be a collection of metrics $d_t: M_t^2 \rightarrow R$. Let M_t be a nonempty set for all $t \in T$. Let $i: H_d \rightarrow T$ be a mapping defined by $i(x) = x$. Let $f \in \mathcal{M}(T)$. Then $\mathcal{T}_S = \mathcal{T}_f$ if and only if $f \circ i_*$ is continuous.

Proof. Necessity follows by 3.4. Sufficiency follows by 3.2.

3.7. Example. Let $d = (d_n)_{n \in N}$ be a collection of metrics $d_n: \langle 0, 1/n \rangle^2 \rightarrow R$, $d_n(u, v) = |u - v|$ for each $u, v \in \langle 0, 1/n \rangle$, where N is the set of all positive integer numbers. Let $i: H_d \rightarrow N$, $i(x) = x$ (therefore i is the identity, since $H_d = N$).

Let $f: N^+ \rightarrow R$ be a function defined by $f(x) = \sup_{n \in N} (\min(1, x(n)))$ for all $x \in N^+$.

Then we can verify that $f \in \mathcal{M}(N)$, $\mathcal{T}_S = \mathcal{T}_f$ but $f \circ i_*$ is not continuous.

3.8. Example. Let $d = (d_n)_{n \in N}$ be a collection of metrics $d_n: \{0, 1\}^2 \rightarrow R$, $d_n(0, 1) = 1$ for all $n \in N$. Let $i: H_d \rightarrow N$, $i(x) = x$ (since $H_d = \emptyset$, i is the empty mapping). Let $f: N^+ \rightarrow R$ be a function defined by $f(\Theta) = 0$ and $f(x) = 1 \forall x \in N^+, x \neq \Theta$. Then we can show that $f \in \mathcal{M}(N)$, $f \circ i_*$ is continuous but $\mathcal{T}_S \neq \mathcal{T}_f$.

3.9. Proposition. Let $d = (d_t)_{t \in T}$ be a collection of metrics $d_t: M_t^2 \rightarrow R$. Let $E, H_d \subset E \subset I_d$, be such a set that $T - E$ is a finite set. Let $f \in \mathcal{M}(T)$ be a mapping such that $\forall \varepsilon > 0 \exists c \in T^+$:

(i) $\exists F \subset E$ finite $\forall t \in E - F: c(t) \geq \sup \text{Im } d_t$,

(ii) $\forall t \in E: c(t) > 0$,

(iii) $f(c) < \varepsilon$.

Then $\mathcal{T}_S = \mathcal{T}_f$.

Proof. Let $x \in \prod_{t \in T} M_t$ and $\varepsilon > 0$. Since $T - E$ is a finite set, there exists $\delta > 0$ such that $\forall t \in T - E \forall y \in M_t, y \neq x(t): d_t(x(t), y) \geq \delta$. Further, since $\varepsilon/2 > 0$, there exists $c \in T^+$ such that $(\forall t \in E: c(t) > 0)$ & $(\exists F \subset E$ finite $\forall t \in E - F: c(t) \geq \sup \text{Im } d_t)$ & $(f(c) < \varepsilon/2)$. Since $F \subset E$, we have $\forall t \in F: c(t) > 0$. Since F is a finite set, there exists $\gamma > 0$ such that $\forall t \in F: c(t) \geq \gamma$. Let K be a nonempty finite set such that $((T - E) \cup F) \subset K \subset T$. Put $V = \bigcap_{t \in K} \pi_t^{-1}(S(x(t), \min(\gamma, \delta)))$. Let $y \in V$. Then $\forall t \in E - F: d_t(x(t), y(t)) \leq \sup \text{Im } d_t \leq c(t), \forall t \in T - E: d_t(x(t), y(t)) = 0 \leq c(t), \forall t \in F: d_t(x(t), y(t)) \leq \gamma \leq c(t)$, i.e. $\varrho_d(x, y) \leq c$. Then $\varrho_d(x, y) \leq 2c$, hence $(f \circ \varrho_d)(x, y) = f(\varrho_d(x, y)) \leq 2f(c) < 2\varepsilon/2 = \varepsilon$, i.e. $y \in S(x, \varepsilon)$. Therefore $x \in V \subset S(x, \varepsilon), V \in \mathcal{T}_S$. Then $\mathcal{T}_f \subset \mathcal{T}_S$.

3.10. Corollary. Let $d = (d_i)_{i \in T}$ be a collection of metrics $d_i: M_i^2 \rightarrow R$. Let $\forall \varepsilon > 0$ $\exists H \subset T$ finite $\forall t \in T - H: \sup \text{Im } d_t < \varepsilon$. Let $E, H_d \subset E \subset T$, be such a set that $T - E$ is a finite set. Let $f \in \mathcal{M}(T)$ be a mapping such that

$$\forall \varepsilon > 0 \exists \gamma > 0 \exists c \in T^+ : (f(c) < \varepsilon) \& (\forall t \in E: c(t) \geq \gamma).$$

Then $\mathcal{T}_S = \mathcal{T}_f$.

Proof. Let $\varepsilon > 0$. Then $\exists \gamma > 0 \exists c \in T^+ : (f(c) < \varepsilon) \& (\forall t \in E: c(t) \geq \gamma)$. Then $\exists H \subset T$ finite $\forall t \in T - H: \sup \text{Im } d_t < \gamma$. Put $F = H \cap E$. Then $F \subset E$, F is a finite set and $\forall t \in E - F: c(t) \geq \gamma > \sup \text{Im } d_t$. Therefore $\mathcal{T}_S = \mathcal{T}_f$ by 3.9.

3.11. Example. Let $d = (d_n)_{n \in N}$ be a collection of metrics $d_n: \langle 0, 1/n^2 \rangle^2 \rightarrow R$, $d_n(u, v) = |u - v| \forall u, v \in \langle 0, 1/n^2 \rangle$. Let $f: N^+ \rightarrow R$ be a function defined by $f(x) = \sup_{n \in N} (\min(1, n \cdot x(n)))$ for each $x \in N^+$. Then by 2.6 $f \in \mathcal{M}(N)$, by 3.9 $\mathcal{T}_S = \mathcal{T}_f$, but d and f do not satisfy the hypothesis of 3.10.

3.12. Theorem. Let $d = (d_i)_{i \in T}$ be a collection of metrics $d_i: M_i^2 \rightarrow R$. Let M_i be a nonempty set for all t in T . Let $f \in \mathcal{M}(T)$. Then $\mathcal{T}_S = \mathcal{T}_f$ if and only if

$$\forall \varepsilon > 0 \exists F \subset T \text{ finite } \exists \delta > 0 \forall \alpha \in N^{(T - (I_d \cup F))} \exists a \in T^+ :$$

- (i) $\forall t \in (T - (I_d \cup F)): a(t) \geq \alpha(t)$,
- (ii) $\forall t \in (I_d - F): a(t) \geq \sup \text{Im } d_t$,
- (iii) $\forall t \in (F \cap H_d): a(t) \geq \delta$,
- (iv) $f(a) < \varepsilon$.

Proof. Necessity. Let $t \in H_d$. Then $\exists x_t \in M_t \forall \varepsilon > 0 \exists y \in M_t: 0 < d_t(x_t, y) < \varepsilon$. Since $\forall t \in T: M_t \neq \emptyset$, we have $\forall t \in (T - H_d) \exists x_t \in M_t$. Define a mapping $x: T \rightarrow \bigcup_{t \in T} M_t$ by $x(t) = x_t$ for all $t \in T$. Let $\varepsilon > 0$. Since $\mathcal{T}_S = \mathcal{T}_f$, $S(x, \varepsilon/4) \in \mathcal{T}_S$. Therefore

$$\exists F \subset T, F \neq \emptyset \text{ finite } \exists \gamma > 0: \bigcap_{t \in F} \pi_t^{-1}(S(x(t), \gamma)) \subset S(x, \varepsilon/4).$$

Let $t \in F \cap H_d$. Then $\exists y_t \in M_t: 0 < d_t(x(t), y_t) < \gamma$. If $F \cap H_d \neq \emptyset$ put $\delta = \min_{t \in F \cap H_d} d_t(x(t), y_t) > 0$. If $F \cap H_d = \emptyset$, put $\delta = 1$. Let $\alpha \in N^{(T - (I_d \cup F))}$ Let $t \in T - (I_d \cup F)$. Then $\exists y_t \in M_t: d_t(x(t), y_t) \geq \alpha(t)$. Let $t \in I_d - F$. If $\sup \text{Im } d_t > 0$, there exists $y_t \in M_t$:

$$d_t(x(t), y_t) > (1/4) \cdot \sup \text{Im } d_t.$$

If $\sup \text{Im } d_t = 0$, put $y_t = x(t)$. Put $y_t = x(t)$ for each $t \in F - H_d$.

Define a mapping $y: T \rightarrow \bigcup_{t \in T} M_t$ by $y(t) = y_t$ for all $t \in T$. Put $a = 4\varrho_d(x, y)$. Then

$$f(a) \leq 4 \cdot f(\varrho_d(x, y)) = 4 \cdot (f \circ \varrho_d)(x, y) < 4 \cdot \varepsilon/4 = \varepsilon.$$

Sufficiency. Let $x \in \prod_{t \in T} M_t$ and $\varepsilon > 0$. Since $\varepsilon/2 > 0$, there exists a finite set $F \subset T$ such that

$$\begin{aligned} \exists \delta > 0 \forall \alpha \in N^{(T - (I_d \cup F))} \exists a \in T^+ : (\forall t \in (T - (I_d \cup F)) : \\ a(t) \geq \alpha(t)) \& (\forall t \in I_d - F : a(t) \geq \sup \text{Im } d_t) \& \\ \& (\forall t \in F \cap H_d : a(t) \geq \delta) \& (f(a) < \varepsilon/2). \end{aligned}$$

Since $F - H_d$ is a finite set

$$\exists \gamma > 0 \forall t \in (F - H_d) \forall y \in M_t, y \neq x(t) : d_t(x(t), y) \geq \gamma.$$

Let $K, F \subset K \subset T$, be a nonempty finite set.

Put $V = \bigcap_{t \in K} \pi_t^{-1}(S(x(t), \min(\gamma, \delta)))$. Let $y \in V$. Let $t \in (T - (I_d \cup F))$. Then there exists a positive integer n_t such that $d_t(x(t), y(t)) \leq n_t$. Define a mapping $\alpha: (T - (I_d \cup F)) \rightarrow N$ by $\alpha(t) = n_t$ for each $t \in T - (I_d \cup F)$. Then $\exists a \in T^+ : (\forall t \in (T - (I_d \cup F)) : a(t) \geq \alpha(t)) \& (\forall t \in I_d - F : a(t) \geq \sup \text{Im } d_t) \& (\forall t \in F : a(t) \geq \delta) \& (f(a) < \varepsilon/2)$.

Then $\forall t \in I_d - F : d_t(x(t), y(t)) \leq \sup \text{Im } d_t \leq a(t), \forall t \in F \cap H_d : d_t(x(t), y(t)) < \delta \leq a(t), \forall t \in (T - (I_d \cup F)) : d_t(x(t), y(t)) \leq a(t) \leq a(t), \forall t \in F - H_d : d_t(x(t), y(t)) = 0 \leq a(t)$, i.e. $\varrho_a(x, y) \leq a$. Then $\varrho_a(x, y) \leq 2a$, hence $(f \circ \varrho_a)(x, y) = f(\varrho_a(x, y)) \leq 2f(a) < 2 \cdot \varepsilon/2 = \varepsilon$, i.e. $y \in S(x, \varepsilon)$. Therefore $x \in V \subset S(x, \varepsilon)$, $V \in \mathcal{T}_s$. Then $\mathcal{T}_f \subset \mathcal{T}_s$.

3.13.Example. Let d be a collection of metrics from example 3.8. Define a function $k: N^+ - \{\Theta\} \rightarrow N$ by $k(x) = \min \{n \in N : x(n) \neq 0\}$. Define a function $f: N^+ \rightarrow R$ by $f(x) = e^{-k(x)}$ for $x \in N^+, x \neq \Theta$ and $f(\Theta) = 0$. Then by 2.6 $f \in \mathcal{M}(N)$, by 3.12 $\mathcal{T}_s = \mathcal{T}_f$, but f and d do not satisfy the hypotheses of either 3.2 or 3.9.

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О ПРОИЗВЕДЕНИИ МЕТРИЧЕСКИХ ПРОСТРАНСТВ

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Резюме

Пусть T является непустым множеством. Обозначим $\mathcal{M}(T)$ множество всех отображений $f: \{x \in R^T; \forall t \in T: x(t) \geq 0\} \rightarrow R$, для которых

$$(1) \quad d(x, y) = f(\{d_t(x, y)\}_{t \in T})$$

является метрикой на множестве

$$\prod_{t \in T} M_t$$

для каждого семейства метрических пространств $\{(M_t, d_t)\}_{t \in T}$. В этой работе мы предлагаем характеристику множества $\mathcal{M}(T)$, а также необходимое и достаточное условие метризации топологии произведения на

$$\prod_{t \in T} M_t$$

при помощи метрики (1).